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Global existence for compressible Navier–Stokes–Poisson equations in three and higher dimensions

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ABSTRACT

The compressible Navier–Stokes–Poisson system is concerned in the present paper, and the global existence and uniqueness of the strong solution is shown in the framework of hybrid Besov spaces in three and higher dimensions.

1. Introduction

In the present paper, we consider the Cauchy problem of the following compressible Navier–Stokes–Poisson equations

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P(\rho) = \rho \nabla \phi + \mu \Delta \mathbf{u} + (\mu + \lambda) \nabla \operatorname{div} \mathbf{u}, \\ \Delta \phi = \rho - \bar{\rho}, \\ (\rho, \mathbf{u})(0) = (\rho_0, \mathbf{u}_0), \end{cases} \tag{1.1}$$

for $(t, x) \in [0, +\infty) \times \mathbb{R}^N$, $N \geqslant 3$. ρ, $\mathbf{u}$ and ϕ denote the electron density, electron velocity and the electrostatic potential, respectively. $P(\rho) = \frac{1}{2}\rho^\gamma$ is the pressure with $\gamma = 2$. μ, λ are the constant viscosity coefficients satisfying $\mu > 0$ and $2\mu + N\lambda \geqslant 0$. The constant $\bar{\rho}$ stands for the density of

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positively charged background ions. The Navier–Stokes–Poisson system is a simplified model (for instance, the energy equation is not taken into granted) to describe the dynamics of a charge transport where the compressible charged fluid interacts with its own electric field against a charged ion background [5].

Recently, many interesting researches have been devoted to many topics of the compressible Navier–Stokes–Poisson (NSP) system. The global existence of weak solutions of the compressible NSP system subject to large initial data is shown [7,18]. The quasi-neutral limits and related combining asymptotical limits are proven [6,8,12,17]. In the case that the potential force representing the self-gravity in stellar gases, the global existence of weak solutions and asymptotical behaviors are also investigated recently, and the stability analysis for compressible Navier–Stokes–Poisson and related systems is also carried out, refer for instance to [9–11,14] and references therein.

The global existence of the classical solution is shown recently [13] in terms of the framework by Matsumura and Nishida. In addition, the influence of the electric field is justified, which affects the dissipation of the viscosity and the time-decay rate of global solutions of IVP (1.1) to the equilibrium state $(\bar\rho, 0)$, namely,

$$c_1(1+t)^{-\frac{3}{4}} \leqslant \left\|(\rho-\bar\rho)(t)\right\|_{L^2(\mathbb{R}^3)} \leqslant C(1+t)^{-\frac{3}{4}}, \tag{1.2}$$

$$c_1(1+t)^{-\frac{1}{4}} \leqslant \left\|m(t)\right\|_{L^2(\mathbb{R}^3)} \leqslant C(1+t)^{-\frac{1}{4}}, \tag{1.3}$$

where the decay rate of the momentum or the velocity is slower than the rate $(1+t)^{-\frac{3}{4}}$ for compressible Navier–Stokes equations. A natural question follows then, that is, whether the similar phenomena can be shown for global weak solutions or strong solutions with lower regularity.

To this end, the first step is to show the global existence of strong solutions in some Besov space with lower regularity. In this paper, with the help of the classical Friedrichs' regularization method, Littlewood–Paley analysis and hybrid Besov spaces, we are able to construct the approximate solutions, obtain the a priori estimates in hybrid Besov spaces, and prove the global existence of the unique strong solution by the compactness arguments as in [1,3,15]. Indeed, in terms of the div-curl decomposition we can decompose the velocity vector field into a vector field of the compressible part and an incompressible part. Then, the original compressible system for the density and the velocity can be decoupled into a system involving only the compressible system for the ir-rotational (compressible) part of velocity vector field and the electron density and the diffusion equation for the divergence free (incompressible) part of velocity vector field as used in [3]. Thus, we can investigate the compressible velocity field part and the incompressible velocity field part separately to get the expected estimates in some hybrid Besov spaces. As one can see later, however, the appearance of the electric field leads to the rotational coupling effect and the loss of regularity of density and the velocity vector field.

For simplicity, we only deal with the case $\gamma = 2$, the arguments used here can be applied to show the global existence for general $\gamma > 1$. We have the main theorem as follows.

Theorem 1.1. *Let $N \geqslant 3$, $\gamma = 2$, $\mu > 0$ and $2\mu + N\lambda \geqslant 0$. Assume $\rho_0 - \bar\rho \in \tilde B_{2,1}^{\frac{N}{2}-\frac{5}{2},\frac{N}{2}}$ and $\mathbf{u}_0 \in \tilde B_{2,1}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-1}$. Then, there exist two positive constants α small enough and M such that if*

$$\|\rho_0-\bar\rho\|_{\tilde B_{2,1}^{\frac{N}{2}-\frac{5}{2},\frac{N}{2}}} + \|\mathbf{u}_0\|_{\tilde B_{2,1}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-1}} \leqslant \alpha,$$

then (1.1) *yields a unique global solution $(\rho, \mathbf{u}, \phi)$ such that $(\rho-\bar\rho, \mathbf{u}, \phi)$ belongs to*

$$\begin{aligned} E := {}& \mathcal{C}\big(\mathbb{R}^+; \tilde B_{2,1}^{\frac{N}{2}-\frac{5}{2},\frac{N}{2}} \times \big(\tilde B_{2,1}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-1}\big)^N \times \tilde B_{2,1}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+2}\big) \\ & \cap L^1\big(\mathbb{R}^+; \tilde B_{2,1}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}} \times \big(\tilde B_{2,1}^{\frac{N}{2}+\frac{1}{2},\frac{N}{2}+1}\big)^N \times \tilde B_{2,1}^{\frac{N}{2}+\frac{3}{2},\frac{N}{2}+2}\big), \end{aligned}$$

and satisfies

$$\|(\rho-\bar\rho,\mathbf{u},\phi)\|_E \leqslant M\big(\|\rho_0-\bar\rho\|_{\tilde B_{2,1}^{\frac N2-\frac52,\frac N2}}+\|\mathbf{u}_0\|_{\tilde B_{2,1}^{\frac N2-\frac32,\frac N2-1}}\big),$$

where M is independent of the initial data and the hybrid space $\tilde B_{2,1}^{s_1,s_2}=\dot B_{2,1}^{s_1}\cap\dot B_{2,1}^{s_2}$ for $s_1\leqslant s_2$.

The paper is organized as follows. We recall some Littlewood–Paley theories for homogeneous Besov spaces and give the definitions and some properties of hybrid Besov spaces in the second section. In Sections 3–4, we are dedicated into reformulation of the system and proving a priori estimates for a linearized system with convection terms. In Section 5, we prove the global existence and uniqueness of the solution.

2. Littlewood–Paley decomposition and Besov spaces

Let $\psi:\mathbb{R}^N\to[0,1]$ be a radial smooth cut-off function valued in $[0,1]$ such that

$$\psi(\xi)=\begin{cases}1, & |\xi|\leqslant 3/4,\\ \text{smooth}, & 3/4<|\xi|<4/3,\\ 0, & |\xi|\geqslant 4/3.\end{cases}$$

Let $\varphi(\xi)$ be the function

$$\varphi(\xi):=\psi(\xi/2)-\psi(\xi).$$

Thus, ψ is supported in the ball $\{\xi\in\mathbb{R}^N\colon\ |\xi|\leqslant 4/3\}$, and φ is also a smooth cut-off function valued in $[0,1]$ and supported in the annulus $\{\xi\in\mathbb{R}^N\colon\ 3/4\leqslant|\xi|\leqslant 8/3\}$. By construction, we have

$$\sum_{k\in\mathbb{Z}}\varphi\big(2^{-k}\xi\big)=1,\quad \forall\xi\neq 0.$$

One can define the dyadic blocks as follows. For $k\in\mathbb{Z}$, let

$$\triangle_k f:=\mathscr{F}^{-1}\varphi\big(2^{-k}\xi\big)\mathscr{F}f.$$

The formal decomposition

$$f=\sum_{k\in\mathbb{Z}}\triangle_k f \tag{2.1}$$

is called homogeneous Littlewood–Paley decomposition. Actually, this decomposition works for just about any locally integrable function which has some decay at infinity, and one usually has all the convergence properties of the summation that one needs. Thus, the r.h.s. of (2.1) does not necessarily converge in $\mathscr{S}'(\mathbb{R}^N)$. Even if it does, the equality is not always true in $\mathscr{S}'(\mathbb{R}^N)$. For instance, if $f\equiv 1$, then all the projections $\triangle_k f$ vanish. Nevertheless, (2.1) is true modulo polynomials, in other words (cf. [4,16]), if $f\in\mathscr{S}'(\mathbb{R}^N)$, then $\sum_{k\in\mathbb{Z}}\triangle_k f$ converges modulo $\mathscr{P}[\mathbb{R}^N]$ and (2.1) holds in $\mathscr{S}'(\mathbb{R}^N)/\mathscr{P}[\mathbb{R}^N]$.

Definition 2.1. Let $s\in\mathbb{R}$, $1\leqslant p,q\leqslant\infty$. For $f\in\mathscr{S}'(\mathbb{R}^N)$, we write

$$\|f\|_{\dot B_{2,1}^s}=\sum_{k\in\mathbb{Z}}2^{ks}\|\triangle_k f\|_{L^2}.$$

A difficulty comes from the choice of homogeneous spaces at this point. Indeed, $\|\cdot\|_{\dot{B}^s_{2,1}}$ cannot be a norm on $\{f \in \mathscr{S}'(\mathbb{R}^N)\colon \|f\|_{\dot{B}^s_{2,1}} < \infty\}$ because $\|f\|_{\dot{B}^s_{2,1}} = 0$ means that f is a polynomial. This enforces us to adopt the following definition for homogeneous Besov spaces (cf. [3]).

Definition 2.2. Let $s \in \mathbb{R}$ and $m = -[\frac{N}{2} + 1 - s]$. If $m < 0$, then we define $\dot{B}^s_{2,1}(\mathbb{R}^N)$ as

$$\dot{B}^s_{2,1} = \left\{ f \in \mathscr{S}'(\mathbb{R}^N)\colon \ \|f\|_{\dot{B}^s_{2,1}} < \infty \text{ and } u = \sum_{k\in\mathbb{Z}} \triangle_k f \text{ in } \mathscr{S}'(\mathbb{R}^2) \right\}.$$

If $m \geqslant 0$, we denote by $\mathscr{P}_m$ the set of two variables polynomials of degree less than or equal to m and define

$$\dot{B}^s_{2,1} = \left\{ f \in \mathscr{S}'(\mathbb{R}^N)/\mathscr{P}_m\colon \ \|f\|_{\dot{B}^s_{2,1}} < \infty \text{ and } u = \sum_{k\in\mathbb{Z}} \triangle_k f \text{ in } \mathscr{S}'(\mathbb{R}^N)/\mathscr{P}_m \right\}.$$

For the composition of functions, we have the following estimates.

Lemma 2.3. *(See [3, Lemma 2.7].) Let $s > 0$ and $u \in \dot{B}^s_{2,1} \cap L^\infty$. Then, it holds:*

(i) *Let $F \in W^{[s]+2,\infty}_{\mathrm{loc}}(\mathbb{R}^N)$ with $F(0) = 0$. Then $F(u) \in \dot{B}^s_{2,1}$. Moreover, there exists a function of one variable C_0 depending only on s and F, and such that*

$$\left\|F(u)\right\|_{\dot{B}^s_{2,1}} \leqslant C_0\left(\|u\|_{L^\infty}\right)\|u\|_{\dot{B}^s_{2,1}}.$$

(ii) *If $u, v \in \dot{B}^{\frac{N}{2}}_{2,1}$, $(v-u) \in \dot{B}^s_{2,1}$ for $s \in (-\frac{N}{2}, \frac{N}{2}]$ and $G \in W^{[\frac{N}{2}]+3,\infty}_{\mathrm{loc}}(\mathbb{R}^N)$ satisfies $G'(0) = 0$, then $G(v) - G(u) \in \dot{B}^s_{2,1}$ and there exists a function of two variables C depending only on s, N and G, and such that*

$$\left\|G(v) - G(u)\right\|_{\dot{B}^s_{2,1}} \leqslant C\left(\|u\|_{L^\infty}, \|v\|_{L^\infty}\right)\left(\|u\|_{\dot{B}^{\frac{N}{2}}_{2,1}} + \|v\|_{\dot{B}^{\frac{N}{2}}_{2,1}}\right)\|v-u\|_{\dot{B}^s_{2,1}}.$$

We also need hybrid Besov spaces for which regularity assumptions are different in low frequencies and high frequencies [3]. We are going to recall the definition of these new spaces and some of their main properties.

Definition 2.4. Let $s, t \in \mathbb{R}$. We define

$$\|f\|_{\tilde{B}^{s,t}_{2,1}} = \sum_{k\leqslant 0} 2^{ks}\|\triangle_k f\|_{L^2} + \sum_{k>0} 2^{kt}\|\triangle_k f\|_{L^2}.$$

Let $m = -[\frac{N}{2} + 1 - s]$, we then define

$$\tilde{B}^{s,t}_{2,1}(\mathbb{R}^N) = \left\{ f \in \mathscr{S}'(\mathbb{R}^N)\colon \ \|f\|_{\tilde{B}^{s,t}_{2,1}} < \infty \right\}, \quad \text{if } m < 0,$$
$$\tilde{B}^{s,t}_{2,1}(\mathbb{R}^N) = \left\{ f \in \mathscr{S}'(\mathbb{R}^N)/\mathscr{P}_m\colon \ \|f\|_{\tilde{B}^{s,t}_{2,1}} < \infty \right\}, \quad \text{if } m \geqslant 0.$$

Lemma 2.5. *We have the following inclusions for hybrid Besov spaces.*

(i) *We have $\tilde{B}^{s,s}_{2,1} = \dot{B}^s_{2,1}$.*

(ii) *If* $s \leqslant t$ *then* $\tilde{B}^{s,t}_{2,1} = \dot{B}^{s}_{2,1} \cap \dot{B}^{t}_{2,1}$. *Otherwise,* $\tilde{B}^{s,t}_{2,1} = \dot{B}^{s}_{2,1} + \dot{B}^{t}_{2,1}$.
(iii) *The space* $\tilde{B}^{0,s}_{2,1}$ *coincides with the usual inhomogeneous Besov space* $B^{s}_{2,1}$.
(iv) *If* $s_1 \leqslant s_2$ *and* $t_1 \geqslant t_2$, *then* $\tilde{B}^{s_1,t_1}_{2,1} \hookrightarrow \tilde{B}^{s_2,t_2}_{2,1}$.

Let us now recall some useful estimates for the product in hybrid Besov spaces.

Lemma 2.6. *(See [3, Proposition 2.10].) Let* $s_1, s_2 > 0$ *and* $f, g \in L^\infty \cap \tilde{B}^{s_1,s_2}_{2,1}$. *Then* $fg \in \tilde{B}^{s_1,s_2}_{2,1}$ *and*

$$\|fg\|_{\tilde{B}^{s_1,s_2}_{2,1}} \lesssim \|f\|_{L^\infty}\|g\|_{\tilde{B}^{s_1,s_2}_{2,1}} + \|f\|_{\tilde{B}^{s_1,s_2}_{2,1}}\|g\|_{L^\infty}.$$

Let $s_1, s_2, t_1, t_2 \leqslant \frac{N}{2}$ *such that* $\min(s_1+s_2, t_1+t_2) > 0$, $f \in \tilde{B}^{s_1,t_1}_{2,1}$ *and* $g \in \tilde{B}^{s_2,t_2}_{2,1}$. *Then* $fg \in \tilde{B}^{s_1+s_2-1,t_1+t_2-1}_{2,1}$ *and*

$$\|fg\|_{\tilde{B}^{s_1+s_2-\frac{N}{2},t_1+t_2-\frac{N}{2}}_{2,1}} \lesssim \|f\|_{\tilde{B}^{s_1,t_1}_{2,1}}\|g\|_{\tilde{B}^{s_2,t_2}_{2,1}}.$$

For $\alpha, \beta \in \mathbb{R}$, let us define the following characteristic function on $\mathbb{Z}$:

$$\tilde{\varphi}^{\alpha,\beta}(r) = \begin{cases} \alpha, & \text{if } r \leqslant 0, \\ \beta, & \text{if } r \geqslant 1. \end{cases}$$

Then, we can recall the following lemma.

Lemma 2.7. *(See [3, Lemma 6.2].) Let F be a homogeneous smooth function of degree m. Suppose that* $-N/2 < s_1, t_1, s_2, t_2 \leqslant 1 + N/2$. *The following two estimates hold*:

$$\big|\big(F(D)\triangle_k(\mathbf{v}\cdot\nabla a), F(D)\triangle_k a\big)\big| \lesssim c_k 2^{-k(\tilde{\varphi}^{s_1,s_2}(k)-m)}\|\mathbf{v}\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\|a\|_{\tilde{B}^{s_1,s_2}_{2,1}}\big\|F(D)\triangle_k a\big\|_{L^2},$$

$$\begin{aligned}
&\big|\big(F(D)\triangle_k(\mathbf{v}\cdot\nabla a), \triangle_k b\big) + \big(\triangle_k(\mathbf{v}\cdot\nabla b), F(D)\triangle_k a\big)\big| \\
&\quad\lesssim c_k\|\mathbf{v}\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}} \times \big(2^{-k(\tilde{\varphi}_{t_1,t_2}(k)-m)}\big\|F(D)\triangle_k a\big\|_{L^2}\|b\|_{\tilde{B}^{t_1,t_2}_{2,1}} + 2^{-k(\tilde{\varphi}^{s_1,s_2}(k)-m)}\|a\|_{\tilde{B}^{s_1,s_2}_{2,1}}\|\triangle_k b\|_{L^2}\big),
\end{aligned}$$

where $(\cdot,\cdot)$ *denotes the* L^2*-inner product, the operator* $F(D)$ *is defined by* $F(D)f := \mathscr{F}^{-1}F(\xi)\mathscr{F}f$ *and* $\sum_{k\in\mathbb{Z}} c_k \leqslant 1$.

3. Reformulation of the original system

Let $\tilde{\rho} = \rho - \bar{\rho}$. Then (1.1) can be rewritten as

$$\begin{cases} \tilde{\rho}_t + \mathbf{u}\cdot\nabla\tilde{\rho} + \bar{\rho}\operatorname{div}\mathbf{u} = -\tilde{\rho}\operatorname{div}\mathbf{u}, \\ \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \bar{\rho}^{-1}\mu\Delta\mathbf{u} - \bar{\rho}^{-1}(\mu+\lambda)\nabla\operatorname{div}\mathbf{u} + \nabla\tilde{\rho} - \nabla\phi \\ \quad = -\dfrac{\tilde{\rho}}{\bar{\rho}(\tilde{\rho}+\bar{\rho})}\big(\mu\Delta\mathbf{u} + (\mu+\lambda)\nabla\operatorname{div}\mathbf{u}\big), \\ \Delta\phi = \tilde{\rho}. \end{cases} \tag{3.1}$$

Denote $\Lambda^s z := \mathscr{F}^{-1}|\xi|^s\mathscr{F}z$ for all $s \in \mathbb{R}$. Let $\mathrm{c} = \Lambda^{-1}\operatorname{div}\mathbf{u}$ be the “compressible part” of the velocity and $\mathbf{I} = \Lambda^{-1}\operatorname{curl}\mathbf{u}$ be the “incompressible part.” Then, we have

$$\mathbf{u} = -\Lambda^{-1}\nabla\mathrm{c} - \Lambda^{-1}\operatorname{div}\mathbf{I},$$

since $\operatorname{div}\operatorname{div}\mathbf{I}=0$. In fact,

$$\operatorname{div}\mathbf{I}=\Lambda^{-1}\operatorname{div}\operatorname{curl}\mathbf{u}=\Lambda^{-1}(\Delta\mathbf{u}-\nabla\operatorname{div}\mathbf{u}),$$

which yields

$$\operatorname{div}\operatorname{div}\mathbf{I}=\Lambda^{-1}\operatorname{div}(\Delta\mathbf{u}-\nabla\operatorname{div}\mathbf{u})=\Lambda^{-1}(\operatorname{div}\Delta\mathbf{u}-\Delta\operatorname{div}\mathbf{u})=0.$$

Moreover,

$$\operatorname{curl}\operatorname{div}\mathbf{I}=\Delta\mathbf{I}.$$

The first equation in (3.1) is changed into

$$\tilde\rho_t+\mathbf{u}\cdot\nabla\tilde\rho+\bar\rho\Lambda\mathrm{c}=-\tilde\rho\operatorname{div}\mathbf{u}. \tag{3.2}$$

For the second equation in (3.1), applying $\Lambda^{-1}\operatorname{div}$ and $\Lambda^{-1}\operatorname{curl}$ to both sides, respectively, we get

$$\begin{cases}\mathrm{c}_t-\bar\rho^{-1}(2\mu+\lambda)\Delta\mathrm{c}-\Lambda\tilde\rho-\Lambda^{-1}\tilde\rho\\ \quad=-\Lambda^{-1}\operatorname{div}\Big\{\mathbf{u}\cdot\nabla\mathbf{u}+\dfrac{\tilde\rho}{\bar\rho(\tilde\rho+\bar\rho)}\big(\mu\Delta\mathbf{u}+(\mu+\lambda)\nabla\operatorname{div}\mathbf{u}\big)\Big\},\\ \mathbf{I}_t-\bar\rho^{-1}\mu\Delta\mathbf{I}=-\Lambda^{-1}\operatorname{curl}\Big\{\mathbf{u}\cdot\nabla\mathbf{u}+\dfrac{\tilde\rho}{\bar\rho(\tilde\rho+\bar\rho)}\big(\mu\Delta\mathbf{u}+(\mu+\lambda)\nabla\operatorname{div}\mathbf{u}\big)\Big\},\end{cases} \tag{3.3}$$

where we have used the fact

$$\operatorname{curl}\nabla f=(\partial_j\partial_i f-\partial_i\partial_j f)_{ij}=(0)_{ij}=\mathbf{0}$$

for any function f.

Because the first equation of (3.3) involves $\Lambda^{-1}\tilde\rho$, we denote $h=\Lambda^{-1}\tilde\rho$. Then, we have

$$\begin{cases}h_t+\Lambda^{-1}(\mathbf{u}\cdot\nabla\Lambda h)+\bar\rho\mathrm{c}=F,\\ \mathrm{c}_t+\mathbf{u}\cdot\nabla\mathrm{c}-\bar\rho^{-1}(2\mu+\lambda)\Delta\mathrm{c}-\Lambda^2h-h=G,\\ \mathbf{I}_t-\bar\rho^{-1}\mu\Delta\mathbf{I}=H,\\ \mathbf{u}=-\Lambda^{-1}\nabla\mathrm{c}-\Lambda^{-1}\operatorname{div}\mathbf{I},\end{cases} \tag{3.4}$$

where

$$F=-\Lambda^{-1}(\Lambda h\operatorname{div}\mathbf{u}),\qquad G=\mathbf{u}\cdot\nabla\mathrm{c}-\Lambda^{-1}\operatorname{div}J,\qquad H=-\Lambda^{-1}\operatorname{curl}J,$$
$$J=\mathbf{u}\cdot\nabla\mathbf{u}+\frac{\tilde\rho}{\bar\rho(\tilde\rho+\bar\rho)}\big(\mu\Delta\mathbf{u}+(\mu+\lambda)\nabla\operatorname{div}\mathbf{u}\big).$$

The third equation is, up to nonlinear terms, a mere heat equation on $\mathbf{I}$. We therefore expect to get appropriate estimates for the incompressible part of the velocity via the following lemma.

Lemma 3.1. *Let $s \in \mathbb{R}$, $r \in [1, +\infty]$, and u solve*

$$\begin{cases} u_t - \bar{\rho}^{-1}\mu\Delta u = f, \\ u(0) = u_0. \end{cases}$$

Then there exists $C > 0$ depending only on N, $\bar{\rho}^{-1}\mu$ and r such that, for all $0 < T \leqslant +\infty$,

$$\|u\|_{L^r_T(\dot{B}^{s+2/r}_{2,1})} \leqslant C\big(\|u_0\|_{\dot{B}^s_{2,1}} + \|f\|_{L^1_T(\dot{B}^s_{2,1})}\big).$$

Moreover, $u \in \mathcal{C}([0,T]; \dot{B}^s_{2,1})$.

For the first two equations, which is a linear coupling system, we can use the following proposition.

Proposition 3.2. *Let (h, c) be a solution of*

$$\begin{cases} h_t + \Lambda^{-1}(\mathbf{v}\cdot\nabla\Lambda h) + \bar{\rho}\mathrm{c} = F, \\ \mathrm{c}_t + \mathbf{v}\cdot\nabla\mathrm{c} - \bar{\rho}^{-1}(2\mu+\lambda)\Delta\mathrm{c} - \Lambda^2 h - h = G \end{cases} \tag{3.5}$$

on $[0,T)$, $(3-N)/2 < s \leqslant (N+1)/2$ and $V(t) = \int_0^t \|\mathbf{v}(\tau)\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\,d\tau$. The following estimate holds on $[0,T)$:

$$\begin{aligned} &\|h(t)\|_{\tilde{B}^{s-1,s+\frac{3}{2}}_{2,1}} + \|\mathrm{c}(t)\|_{\tilde{B}^{s-1,s-\frac{1}{2}}_{2,1}} + \int_0^t \big(\|h(\tau)\|_{\tilde{B}^{s+1,s+\frac{3}{2}}_{2,1}} + \|\mathrm{c}(\tau)\|_{\tilde{B}^{s+1,s+\frac{3}{2}}_{2,1}}\big)\,d\tau \\ &\quad\leqslant Ce^{CV(t)}\left(\|h(0)\|_{\tilde{B}^{s-1,s+\frac{3}{2}}_{2,1}} + \|\mathrm{c}(0)\|_{\tilde{B}^{s-1,s-\frac{1}{2}}_{2,1}} + \int_0^t e^{-CV(\tau)}\big(\|F(\tau)\|_{\tilde{B}^{s-1,s+\frac{3}{2}}_{2,1}} + \|G(\tau)\|_{\tilde{B}^{s-1,s-\frac{1}{2}}_{2,1}}\big)\,d\tau\right), \end{aligned}$$

where C depends only on N and s.

Let us define the functional space for $2 - N/2 < s \leqslant N/2 + 1$:

$$\begin{aligned} E^s &= \mathcal{C}\big(\mathbb{R}^+; \tilde{B}^{s-\frac{3}{2},s+1}_{2,1} \times \big(\tilde{B}^{s-\frac{3}{2},s-1}_{2,1}\big)^N\big) \cap L^1\big(\mathbb{R}^+; \big(\tilde{B}^{s+\frac{1}{2},s+1}_{2,1}\big)^{1+N}\big), \\ \big\|(h,\mathbf{u})\big\|_{E^s} &= \|h\|_{L^\infty(\tilde{B}^{s-\frac{3}{2},s+1}_{2,1})} + \|\mathbf{u}\|_{L^\infty(\tilde{B}^{s-\frac{3}{2},s-1}_{2,1})} + \|h\|_{L^1(\tilde{B}^{s+\frac{1}{2},s+1}_{2,1})} + \|\mathbf{u}\|_{L^1(\tilde{B}^{s+\frac{1}{2},s+1}_{2,1})}. \end{aligned} \tag{3.6}$$

When the time variable t describes a finite length interval $[0,T]$, we will denote by E^s_T and $\|\cdot\|_{E^s_T}$ the corresponding spaces and norms.

4. The estimates for the linear model

This section is devoted to the proof of Proposition 3.2. Let (h, c) be a solution of (3.5) and denote $\tilde{f} := e^{-KV(t)}f$ for any function f. Then the system (3.5) can be transformed into the following form:

$$\begin{cases} \tilde{h}_t + \Lambda^{-1}(\mathbf{v}\cdot\nabla\Lambda\tilde{h}) + \bar{\rho}\tilde{\mathrm{c}} = \tilde{F} - KV'(t)\tilde{h}, \\ \tilde{\mathrm{c}}_t + \mathbf{v}\cdot\nabla\tilde{\mathrm{c}} - \bar{\rho}^{-1}(2\mu+\lambda)\Delta\tilde{\mathrm{c}} - \Lambda^2\tilde{h} - \tilde{h} = \tilde{G} - KV'(t)\tilde{\mathrm{c}}. \end{cases} \tag{4.1}$$

Applying the operator $\triangle_k$ to the system (4.1) and denoting $f_k := \triangle_k f$, we have the following system

$$\begin{cases} \partial_t \tilde{h}_k + \Lambda^{-1}(\mathbf{v}\cdot\nabla\Lambda\tilde{h})_k + \bar{\rho}\tilde{c}_k = \tilde{F}_k - KV'(t)\tilde{h}_k, \\ \partial_t \tilde{c}_k + (\mathbf{v}\cdot\nabla\tilde{c})_k - \bar{\rho}^{-1}(2\mu+\lambda)\Delta\tilde{c}_k - \Lambda^2\tilde{h}_k - \tilde{h}_k = \tilde{G}_k - KV'(t)\tilde{c}_k. \end{cases} \tag{4.2}$$

4.1. Low frequencies ($k \leqslant 0$)

Taking the L^2 scalar product of the first equation of (4.2) with $\tilde{h}_k$, of the second equation with $\tilde{c}_k$, we get the following two identities:

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\|\tilde{h}_k\|_{L^2}^2 + \bar{\rho}(\tilde{c}_k,\tilde{h}_k) = (\tilde{F}_k,\tilde{h}_k) - KV'(t)\|\tilde{h}_k\|_{L^2}^2 - \big(\Lambda^{-1}(\mathbf{v}\cdot\nabla\Lambda\tilde{h})_k,\tilde{h}_k\big), \\ \frac{1}{2}\frac{d}{dt}\|\tilde{c}_k\|_{L^2}^2 + \bar{\rho}^{-1}(2\mu+\lambda)\|\Lambda\tilde{c}_k\|_{L^2}^2 - (\Lambda\tilde{h}_k,\Lambda\tilde{c}_k) - (\tilde{h}_k,\tilde{c}_k) \\ \quad = (\tilde{G}_k,\tilde{c}_k) - KV'(t)\|\tilde{c}_k\|_{L^2}^2 - \big((\mathbf{v}\cdot\nabla\tilde{c})_k,\tilde{c}_k\big). \end{cases} \tag{4.3}$$

Now we want to get an equality involving $\Lambda\tilde{h}_k$. To achieve it, we take L^2 scalar product of the first equation of (4.2) with $\Lambda^2\tilde{h}_k$, $\Lambda^4\tilde{h}_k$ and $\Lambda^2\tilde{c}_k$ and of the second equation with $\Lambda^2\tilde{h}_k$ and then sum the last two resulting equalities, which yields, with the Plancherel theorem, that

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\|\Lambda\tilde{h}_k\|_{L^2}^2 + \bar{\rho}(\Lambda\tilde{c}_k,\Lambda\tilde{h}_k) \\ \quad = (\Lambda\tilde{F}_k,\Lambda\tilde{h}_k) - KV'(t)\|\Lambda\tilde{h}_k\|_{L^2}^2 - \big((\mathbf{v}\cdot\nabla\Lambda\tilde{h})_k,\Lambda\tilde{h}_k\big), \\ \frac{d}{dt}\big(\Lambda^2\tilde{h}_k,\tilde{c}_k\big) + \bar{\rho}\|\Lambda\tilde{c}_k\|_{L^2}^2 + \bar{\rho}^{-1}(2\mu+\lambda)\big(\Lambda^2\tilde{h}_k,\Lambda^2\tilde{c}_k\big) - \|\Lambda^2\tilde{h}_k\|_{L^2}^2 - \|\Lambda\tilde{h}_k\|_{L^2}^2 \\ \quad = (\Lambda\tilde{F}_k,\Lambda\tilde{c}_k) - 2KV'(t)\big(\Lambda^2\tilde{h}_k,\tilde{c}_k\big) - \big((\mathbf{v}\cdot\nabla\Lambda\tilde{h})_k,\Lambda\tilde{c}_k\big) + (\Lambda G_k,\Lambda h_k) - \big(\Lambda(\mathbf{v}\cdot\nabla\tilde{c})_k,\Lambda\tilde{h}_k\big). \end{cases} \tag{4.4}$$

A linear combination of (4.3) and (4.4) yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\frac{1}{\bar{\rho}}\|\tilde{h}_k\|_{L^2}^2 + \frac{1}{\bar{\rho}}\|\Lambda\tilde{h}_k\|_{L^2}^2 + \|\tilde{c}_k\|_{L^2}^2 - 2K_1\big(\Lambda^2\tilde{h}_k,\tilde{c}_k\big)\right] + K_1\|\Lambda^2\tilde{h}_k\|_{L^2}^2 + K_1\|\Lambda\tilde{h}_k\|_{L^2}^2 \\ &\quad + \big[\bar{\rho}^{-1}(2\mu+\lambda) - \bar{\rho}K_1\big]\|\Lambda\tilde{c}_k\|_{L^2}^2 - \bar{\rho}^{-1}(2\mu+\lambda)K_1\big(\Lambda^3\tilde{h}_k,\Lambda\tilde{c}_k\big) \\ &= \frac{1}{\bar{\rho}}\big[(\tilde{F}_k,\tilde{h}_k) - KV'(t)\|\tilde{h}_k\|_{L^2}^2 - \big(\Lambda^{-1}(\mathbf{v}\cdot\nabla\Lambda\tilde{h})_k,\tilde{h}_k\big)\big] + \frac{1}{\bar{\rho}}\big[(\Lambda\tilde{F}_k,\Lambda\tilde{h}_k) - KV'(t)\|\Lambda\tilde{h}_k\|_{L^2}^2 \\ &\quad - \big((\mathbf{v}\cdot\nabla\Lambda\tilde{h})_k,\Lambda\tilde{h}_k\big)\big] + (\tilde{G}_k,\tilde{c}_k) - KV'(t)\|\tilde{c}_k\|_{L^2}^2 - \big((\mathbf{v}\cdot\nabla\tilde{c})_k,\tilde{c}_k\big) \\ &\quad - K_1\big[(\Lambda\tilde{F}_k,\Lambda\tilde{c}_k) + (\Lambda G_k,\Lambda h_k) - 2KV'(t)\big(\Lambda^2\tilde{h}_k,\tilde{c}_k\big) - \big((\mathbf{v}\cdot\nabla\Lambda\tilde{h})_k,\Lambda\tilde{c}_k\big) - \big(\Lambda(\mathbf{v}\cdot\nabla\tilde{c})_k,\Lambda\tilde{h}_k\big)\big]. \end{aligned}$$

Noticing that $\|\Lambda\tilde{h}_k\|_{L^2} \leqslant \frac{8}{3}2^k\|\tilde{h}_k\|_{L^2} \leqslant \frac{8}{3}\|\tilde{h}_k\|_{L^2}$ for $k \leqslant 0$, we have

$$\big|\big(\Lambda^2\tilde{h}_k,\tilde{c}_k\big)\big| \leqslant \frac{32M_1}{9}\|\Lambda\tilde{h}_k\|_{L^2}^2 + \frac{1}{2M_1}\|\tilde{c}_k\|_{L^2}^2,$$
$$\big|\big(\Lambda^3\tilde{h}_k,\Lambda\tilde{c}_k\big)\big| \leqslant \frac{32M_2}{9}\|\Lambda^2\tilde{h}_k\|_{L^2}^2 + \frac{1}{2M_2}\|\Lambda\tilde{c}_k\|_{L^2}^2.$$

Thus, we have to choose K_1, M_1 and M_2 satisfying

$$\frac{73}{64} - \frac{32}{9}\bar{\rho}^{-1}(2\mu+\lambda)M_2 > 0, \qquad K_1 < M_1 < \frac{\sqrt{3}}{8\sqrt{\bar{\rho}}},$$
$$2\mu+\lambda - \bar{\rho}^2K_1 - \frac{(2\mu+\lambda)K_1}{2M_2} > 0.$$

Hence, we can take

$$M_1=\frac{1}{4\sqrt{\bar\rho}},\qquad M_2=\frac{5\bar\rho}{16(2\mu+\lambda)},\qquad K_1=\min\left(\frac{\bar\rho(2\mu+\lambda)}{\bar\rho^3+2(2\mu+\lambda)^2},\frac{1}{8\sqrt{\bar\rho}}\right).$$

Denote for $k\leqslant 0$

$$\alpha_k^2:=\frac{1}{\bar\rho}\|\tilde h_k\|_{L^2}^2+\frac{1}{\bar\rho}\|\Lambda\tilde h_k\|_{L^2}^2+\|\tilde c_k\|_{L^2}^2-2K_1\big(\Lambda^2\tilde h_k,\tilde c_k\big).$$

Then, there exist constants c_3 and c_4 such that

$$c_1\alpha_k^2\leqslant\|\tilde h_k\|_{L^2}^2+\|\Lambda\tilde h_k\|_{L^2}^2+\|\tilde c_k\|_{L^2}^2\leqslant c_2\alpha_k^2.$$

Thus, there exists a constant $\hat c$ such that for $k\leqslant 0$

$$\begin{aligned}
&\frac12\frac{d}{dt}\alpha_k^2+\big(\hat c2^{2k}+KV'\big)\alpha_k^2\\
&\quad\leqslant\frac{1}{\bar\rho}\big[(\tilde F_k,\tilde h_k)-\big(\Lambda^{-1}(\mathbf v\cdot\nabla\Lambda\tilde h)_k,\tilde h_k\big)\big]+\frac{1}{\bar\rho}\big[(\Lambda\tilde F_k,\Lambda\tilde h_k)-\big((\mathbf v\cdot\nabla\Lambda\tilde h)_k,\Lambda\tilde h_k\big)\big]\\
&\qquad+(\tilde G_k,\tilde c_k)-\big((\mathbf v\cdot\nabla\tilde c)_k,\tilde c_k\big)-K_1\big[(\Lambda\tilde F_k,\Lambda\tilde c_k)+(\Lambda G_k,\Lambda h_k)\\
&\qquad-\big((\mathbf v\cdot\nabla\Lambda\tilde h)_k,\Lambda\tilde c_k\big)-\big(\Lambda(\mathbf v\cdot\nabla\tilde c)_k,\Lambda\tilde h_k\big)\big].
\end{aligned}\tag{4.5}$$

4.2. *High frequencies* ($k>0$)

Taking the L^2 scalar product of the first equation of (4.2) with $\Lambda\tilde h_k$, of the second equation with $\Lambda\tilde c_k$, we get the following two identities:

$$\begin{cases}
\dfrac12\dfrac{d}{dt}\big\|\Lambda^{\frac12}\tilde h_k\big\|_{L^2}^2+\bar\rho(\tilde c_k,\Lambda\tilde h_k)=(\tilde F_k,\Lambda\tilde h_k)-KV'(t)\big\|\Lambda^{\frac12}\tilde h_k\big\|_{L^2}^2-\big((\mathbf v\cdot\nabla\Lambda\tilde h)_k,\tilde h_k\big),\\
\dfrac12\dfrac{d}{dt}\big\|\Lambda^{\frac12}\tilde c_k\big\|_{L^2}^2+\bar\rho^{-1}(2\mu+\lambda)\big\|\Lambda^{\frac32}\tilde c_k\big\|_{L^2}^2-\big(\Lambda^2\tilde h_k,\Lambda\tilde c_k\big)-(\Lambda\tilde h_k,\tilde c_k)\\
\quad=\big(\Lambda^{\frac12}\tilde G_k,\Lambda^{\frac12}\tilde c_k\big)-KV'(t)\big\|\Lambda^{\frac12}\tilde c_k\big\|_{L^2}^2-\big((\mathbf v\cdot\nabla\tilde c)_k,\Lambda\tilde c_k\big).
\end{cases}\tag{4.6}$$

Now we want to get an equality involving $\Lambda^3\tilde h_k$. To achieve it, we take L^2 scalar product of the first equation of (4.2) with $\Lambda^3\tilde h_k$, $\Lambda^5\tilde h_k$ and $\Lambda^3\tilde c_k$ and of the second equation with $\Lambda^3\tilde h_k$ and then sum the last two resulting equalities, which yields, with the Plancherel theorem, that

$$\begin{cases}
\dfrac12\dfrac{d}{dt}\big\|\Lambda^{\frac32}\tilde h_k\big\|_{L^2}^2+\bar\rho\big(\Lambda\tilde c_k,\Lambda^2\tilde h_k\big)\\
\quad=\big(\Lambda\tilde F_k,\Lambda^2\tilde h_k\big)-KV'(t)\big\|\Lambda^{\frac32}\tilde h_k\big\|_{L^2}^2-\big((\mathbf v\cdot\nabla\Lambda\tilde h)_k,\Lambda^2\tilde h_k\big),\\
\dfrac12\dfrac{d}{dt}\big\|\Lambda^{\frac52}\tilde h_k\big\|_{L^2}^2+\bar\rho\big(\Lambda^2\tilde c_k,\Lambda^3\tilde h_k\big)\\
\quad=\big(\Lambda^2\tilde F_k,\Lambda^3\tilde h_k\big)-KV'(t)\big\|\Lambda^{\frac52}\tilde h_k\big\|_{L^2}^2-\big(\Lambda(\mathbf v\cdot\nabla\Lambda\tilde h)_k,\Lambda^3\tilde h_k\big),\\
\dfrac{d}{dt}\big(\Lambda^3\tilde h_k,\tilde c_k\big)+\bar\rho\big\|\Lambda^{\frac32}\tilde c_k\big\|_{L^2}^2+\bar\rho^{-1}(2\mu+\lambda)\big(\Lambda^3\tilde h_k,\Lambda^2\tilde c_k\big)-\big\|\Lambda^{\frac52}\tilde h_k\big\|_{L^2}^2-\big\|\Lambda^{\frac32}\tilde h_k\big\|_{L^2}^2\\
\quad=\big(\Lambda^2\tilde F_k,\Lambda\tilde c_k\big)-2KV'(t)\big(\Lambda^3\tilde h_k,\tilde c_k\big)-\big((\mathbf v\cdot\nabla\Lambda\tilde h)_k,\Lambda^2\tilde c_k\big)\\
\qquad+\big(\Lambda\tilde G_k,\Lambda^2\tilde h_k\big)-\big(\Lambda(\mathbf v\cdot\nabla\tilde c)_k,\Lambda^2\tilde h_k\big).
\end{cases}\tag{4.7}$$

A linear combination of (4.6) and (4.7) yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\frac{1}{\bar\rho}\big\|\Lambda^{\frac{1}{2}}\tilde h_k\big\|_{L^2}^2+\frac{1}{\bar\rho}\big\|\Lambda^{\frac{3}{2}}\tilde h_k\big\|_{L^2}^2+\bar\rho^{-2}(2\mu+\lambda)K_2\big\|\Lambda^{\frac{5}{2}}\tilde h_k\big\|_{L^2}^2+\big\|\Lambda^{\frac{1}{2}}\tilde c_k\big\|_{L^2}^2-2K_2\big(\Lambda^{\frac{5}{2}}\tilde h_k,\Lambda^{\frac{1}{2}}\tilde c_k\big)\right]\\
&\quad+\big[\bar\rho^{-1}(2\mu+\lambda)-\bar\rho K_2\big]\big\|\Lambda^{\frac{3}{2}}\tilde c_k\big\|_{L^2}^2+K_2\big\|\Lambda^{\frac{5}{2}}\tilde h_k\big\|_{L^2}^2+K_2\big\|\Lambda^{\frac{3}{2}}\tilde h_k\big\|_{L^2}^2\\
&=\frac{1}{\bar\rho}\big[(\tilde F_k,\Lambda\tilde h_k)-KV'(t)\big\|\Lambda^{\frac{1}{2}}\tilde h_k\big\|_{L^2}^2-\big((\mathbf{v}\cdot\nabla\Lambda\tilde h)_k,\tilde h_k\big)\big]\\
&\quad+\frac{1}{\bar\rho}\big[\big(\Lambda\tilde F_k,\Lambda^2\tilde h_k\big)-KV'(t)\big\|\Lambda^{\frac{3}{2}}\tilde h_k\big\|_{L^2}^2-\big((\mathbf{v}\cdot\nabla\Lambda\tilde h)_k,\Lambda^2\tilde h_k\big)\big]\\
&\quad+\bar\rho^{-2}(2\mu+\lambda)K_2\big[\big(\Lambda^2\tilde F_k,\Lambda^3\tilde h_k\big)-KV'(t)\big\|\Lambda^{\frac{5}{2}}\tilde h_k\big\|_{L^2}^2-\big(\Lambda(\mathbf{v}\cdot\nabla\Lambda\tilde h)_k,\Lambda^3\tilde h_k\big)\big]\\
&\quad+\big(\Lambda^{\frac{1}{2}}\tilde G_k,\Lambda^{\frac{1}{2}}\tilde c_k\big)-KV'(t)\big\|\Lambda^{\frac{1}{2}}\tilde c_k\big\|_{L^2}^2-\big(\Lambda^{\frac{1}{2}}(\mathbf{v}\cdot\nabla\tilde c)_k,\Lambda^{\frac{1}{2}}\tilde c_k\big)\\
&\quad-K_2\big[\big(\Lambda^2\tilde F_k,\Lambda\tilde c_k\big)+\big(\Lambda^{\frac{1}{2}}G_k,\Lambda^{\frac{5}{2}}h_k\big)-2KV'(t)\big(\Lambda^{\frac{5}{2}}\tilde h_k,\Lambda^{\frac{1}{2}}\tilde c_k\big)\\
&\quad-\big(\Lambda(\mathbf{v}\cdot\nabla\Lambda\tilde h)_k,\Lambda\tilde c_k\big)-\big(\Lambda(\mathbf{v}\cdot\nabla\tilde c)_k,\Lambda^2\tilde h_k\big)\big].
\end{aligned}$$

Noticing that

$$\big|\big(\Lambda^{\frac{5}{2}}\tilde h_k,\Lambda^{\frac{1}{2}}\tilde c_k\big)\big|\leqslant\frac{M_3}{2}\big\|\Lambda^{\frac{5}{2}}\tilde h_k\big\|_{L^2}^2+\frac{1}{2M_3}\big\|\Lambda^{\frac{1}{2}}\tilde c_k\big\|_{L^2}^2,$$

we have to choose K_2, M_3 such that

$$0<K_2<M_3<\bar\rho^{-2}(2\mu+\lambda).$$

For example, we can take

$$M_3=\frac{2\mu+\lambda}{2\bar\rho^2},\qquad K_2=\frac{2\mu+\lambda}{4\bar\rho^2}.$$

Denote for $k>0$

$$\alpha_k^2:=\frac{1}{\bar\rho}\big\|\Lambda^{\frac{1}{2}}\tilde h_k\big\|_{L^2}^2+\frac{1}{\bar\rho}\big\|\Lambda^{\frac{3}{2}}\tilde h_k\big\|_{L^2}^2+\bar\rho^{-2}(2\mu+\lambda)K_2\big\|\Lambda^{\frac{5}{2}}\tilde h_k\big\|_{L^2}^2+\big\|\Lambda^{\frac{1}{2}}\tilde c_k\big\|_{L^2}^2-2K_2\big(\Lambda^{\frac{5}{2}}\tilde h_k,\Lambda^{\frac{1}{2}}\tilde c_k\big).$$

Then, there are constants c_1 and c_2 such that

$$c_1\alpha_k^2\leqslant\big\|\Lambda^{\frac{1}{2}}\tilde h_k\big\|_{L^2}^2+\big\|\Lambda^{\frac{3}{2}}\tilde h_k\big\|_{L^2}^2+\big\|\Lambda^{\frac{5}{2}}\tilde h_k\big\|_{L^2}^2+\big\|\Lambda^{\frac{1}{2}}\tilde c_k\big\|_{L^2}^2\leqslant c_2\alpha_k^2.$$

Thus, by Bernstein's inequality $\|\Lambda^2\tilde h_k\|_{L^2}\leqslant\frac{8}{3}2^k\|\Lambda\tilde h_k\|_{L^2}$, there exists a constant $\bar c$ such that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\alpha_k^2+(\bar c+KV')\alpha_k^2\\
&\quad\leqslant\frac{1}{\bar\rho}\big[(\tilde F_k,\Lambda\tilde h_k)-\big((\mathbf{v}\cdot\nabla\Lambda\tilde h)_k,\tilde h_k\big)\big]+\frac{1}{\bar\rho}\big[\big(\Lambda\tilde F_k,\Lambda^2\tilde h_k\big)-\big((\mathbf{v}\cdot\nabla\Lambda\tilde h)_k,\Lambda^2\tilde h_k\big)\big]\\
&\qquad+\bar\rho^{-2}(2\mu+\lambda)K_2\big[\big(\Lambda^{\frac{5}{2}}\tilde F_k,\Lambda^{\frac{5}{2}}\tilde h_k\big)-\big(\Lambda^{\frac{3}{2}}(\mathbf{v}\cdot\nabla\Lambda\tilde h)_k,\Lambda^{\frac{5}{2}}\tilde h_k\big)\big]\\
&\qquad+\big(\Lambda^{\frac{1}{2}}\tilde G_k,\Lambda^{\frac{1}{2}}\tilde c_k\big)-\big(\Lambda^{\frac{1}{2}}(\mathbf{v}\cdot\nabla\tilde c)_k,\Lambda^{\frac{1}{2}}\tilde c_k\big)\\
&\qquad-K_2\big[\big(\Lambda^{\frac{5}{2}}\tilde F_k,\Lambda^{\frac{1}{2}}\tilde c_k\big)+\big(\Lambda^{\frac{1}{2}}G_k,\Lambda^{\frac{5}{2}}h_k\big)-\big(\Lambda(\mathbf{v}\cdot\nabla\Lambda\tilde h)_k,\Lambda\tilde c_k\big)-\big(\Lambda(\mathbf{v}\cdot\nabla\tilde c)_k,\Lambda^2\tilde h_k\big)\big].
\end{aligned}\tag{4.8}$$

Now, we combine (4.5) and (4.8). At this stage, we use Lemma 2.7 to estimate the terms involving a convection in (4.5) and (4.8), and eventually get the existence of a sequence $(\gamma_k)_{k\in\mathbb{Z}}$ such that $\sum_{k\in\mathbb{Z}}\gamma_k\leqslant 1$ and

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\alpha_k^2+\big(c\min\big(2^{2k},1\big)+KV'\big)\alpha_k^2\\&\quad\leqslant C\gamma_k\alpha_k 2^{-k(s-1)}\big[\big\|(\tilde F,\tilde G)\big\|_{\tilde B_{2,1}^{s-1,s+\frac{3}{2}}\times\tilde B_{2,1}^{s-1,s-\frac{1}{2}}}+V'\big\|(\tilde h,\tilde{\mathrm{c}})\big\|_{\tilde B_{2,1}^{s-1,s+\frac{3}{2}}\times\tilde B_{2,1}^{s-1,s-\frac{1}{2}}}\big],\end{aligned}\tag{4.9}$$

where $c=\min(\bar c,\hat c)$.

We are going to show that inequality (4.9) provides us with a decay for h and c. We actually have a parabolic decay for c.

4.3. The damping effect for h

Let $\delta>0$ be a small parameter (which will tend to 0) and denote $\beta_k^2=\alpha_k^2+\delta^2$. From (4.9) and dividing by β_k, we get

$$\begin{aligned}&\frac{d}{dt}\beta_k+\big(c\min\big(2^{2k},1\big)+KV'\big)\beta_k\\&\quad\leqslant C\gamma_k 2^{-k(s-1)}\big[\big\|(\tilde F,\tilde G)\big\|_{\tilde B_{2,1}^{s-1,s+\frac{3}{2}}\times\tilde B_{2,1}^{s-1,s-\frac{1}{2}}}+V'\big\|(\tilde h,\tilde{\mathrm{c}})\big\|_{\tilde B_{2,1}^{s-1,s+\frac{3}{2}}\times\tilde B_{2,1}^{s-1,s-\frac{1}{2}}}\big]\\&\qquad+\delta\big(c\min\big(2^{2k},1\big)+KV'\big).\end{aligned}\tag{4.10}$$

Integrating over $[0,t]$ and making δ tend to 0, we have

$$\begin{aligned}&\alpha_k(t)+c\min\big(2^{2k},1\big)\int_0^t\alpha_k(\tau)\,d\tau\\&\quad\leqslant\alpha_k(0)+C2^{-k(s-1)}\int_0^t\gamma_k(\tau)\big\|(\tilde F,\tilde G)\big\|_{\tilde B_{2,1}^{s-1,s+\frac{3}{2}}\times\tilde B_{2,1}^{s-1,s-\frac{1}{2}}}\,d\tau\\&\qquad+\int_0^t V'(\tau)\big[C2^{-k(s-1)}\gamma_k(\tau)\big\|(\tilde h,\tilde{\mathrm{c}})\big\|_{\tilde B_{2,1}^{s-1,s+\frac{3}{2}}\times\tilde B_{2,1}^{s-1,s-\frac{1}{2}}}-K\alpha_k(\tau)\big]\,d\tau.\end{aligned}\tag{4.11}$$

By the definition of α_k^2, we have for any $k\in\mathbb{Z}$

$$2^{k(s-1)}\alpha_k^2\approx 2^{k(s-1)}\max\big(1,2^{\frac{5}{2}k}\big)\|\tilde h_k\|_{L^2}^2+2^{k(s-1)}\max\big(1,2^{\frac{k}{2}}\big)\|\tilde{\mathrm{c}}_k\|_{L^2}^2.\tag{4.12}$$

Thus, we can take K large enough such that

$$\sum_{k\in\mathbb{Z}}\big[C2^{-k(s-1)}\gamma_k(\tau)\big\|(\tilde h,\tilde{\mathrm{c}})\big\|_{\tilde B_{2,1}^{s-1,s+\frac{3}{2}}\times\tilde B_{2,1}^{s-1,s-\frac{1}{2}}}-K\alpha_k(\tau)\big]\leqslant 0.$$

Multiplying both sides of (4.11) by $2^{k(s-1)}$. According to the last inequality, and due to (4.11) and (4.12), we conclude after summation on k in $\mathbb{Z}$, that

$$\|\tilde{h}(t)\|_{\tilde{B}_{2,1}^{s-1,s+\frac{3}{2}}}+\|\tilde{c}(t)\|_{\tilde{B}_{2,1}^{s-1,s-\frac{1}{2}}}+c\int\|\tilde{h}(\tau)\|_{\tilde{B}_{2,1}^{s+1,s+\frac{3}{2}}}+\sum_{k\in\mathbb{Z}}\int_0^t c2^{k(s-\frac{1}{2})}\min\big(2^{\frac{3k}{2}},1\big)\|\tilde{c}_k(\tau)\|_{L^2}\,d\tau$$
$$\leqslant\big\|\big(\tilde{h}(0),\tilde{c}(0)\big)\big\|_{\tilde{B}_{2,1}^{s-1,s+\frac{3}{2}}\times\tilde{B}_{2,1}^{s-1,s-\frac{1}{2}}}+\int_0^t\|(\tilde{F},\tilde{G})(\tau)\|_{\tilde{B}_{2,1}^{s-1,s+\frac{3}{2}}\times\times\tilde{B}_{2,1}^{s-1,s-\frac{1}{2}}}\,d\tau. \tag{4.13}$$

4.4. *The smoothing effect for* c

Once stated the damping effect for h, it is easy to get the smoothing effect on c. Since (4.13) implies the desired estimate for low frequencies, it suffices to prove it for high frequencies only. We therefore suppose in this part that $k>0$.

Define $\theta_k=\|\tilde{c}_k\|_{L^2}$. By the previous inequalities and using Lemma 2.7, the second equality of (4.3) yields, for a constant $c>0$, that

$$\frac{1}{2}\frac{d}{dt}\theta_k^2+c2^{2k}\theta_k^2\lesssim\theta_k\big(\|\Lambda^2\tilde{h}_k\|_{L^2}+\|\tilde{G}_k\|_{L^2}\big)$$
$$+\theta_k V'(t)\big(C\gamma_k2^{-k(s-1)}\min\big(1,2^{-\frac{k}{2}}\big)\|\tilde{c}\|_{\tilde{B}_{2,1}^{s-1,s-\frac{1}{2}}}-K\theta_k\big).$$

Using $\beta_k^2=\theta_k^2+\delta^2$, integrating over $[0,t]$ and then having δ tend to 0, we infer

$$\theta_k(t)+c2^{2k}\theta_k\leqslant\theta_k(0)+C\int_0^t\|\tilde{G}_k\|_{L^2}\,d\tau+C\int_0^t2^{2k}\|\tilde{h}_k(\tau)\|_{L^2}\,d\tau$$
$$+C\int_0^t V'(\tau)\gamma_k(\tau)2^{-k(s-1)}\min\big(1,2^{-\frac{k}{2}}\big)\|\tilde{c}(\tau)\|_{\tilde{B}_{2,1}^{s-1,s-\frac{1}{2}}}\,d\tau.$$

Therefore, we get

$$\sum_{k>0}2^{k(s-\frac{1}{2})}\|\tilde{c}_k(t)\|_{L^2}+c\int_0^t\sum_{k>0}2^{k(s+\frac{3}{2})}\|\tilde{c}_k(\tau)\|_{L^2}\,d\tau$$
$$\leqslant\|\tilde{c}(0)\|_{\tilde{B}_{2,1}^{s-1,s-\frac{1}{2}}}+C\int_0^t\|\tilde{G}(\tau)\|_{\tilde{B}_{2,1}^{s-1,s-\frac{1}{2}}}\,d\tau$$
$$+C\int_0^t\sum_{k>0}2^{k(s+\frac{3}{2})}\|\tilde{h}_k(\tau)\|_{L^2}\,d\tau+CV(t)\sup_{[0,t]}\|\tilde{c}(t)\|_{\tilde{B}_{2,1}^{s-1,s-\frac{1}{2}}}.$$

Using (4.13), we eventually conclude that

$$c\int_0^t\sum_{l>0}2^{k(s+\frac{3}{2})}\|\tilde{c}_k(\tau)\|_{L^2}\,d\tau$$
$$\leqslant\big(C+CV(t)\big)\left(\|\tilde{h}(0)\|_{\tilde{B}_{2,1}^{s-1,s+\frac{3}{2}}}+\|\tilde{c}(0)\|_{\tilde{B}_{2,1}^{s-1,s-\frac{1}{2}}}+\int_0^t\big(\|\tilde{F}(\tau)\|_{\tilde{B}_{2,1}^{s-1,s+\frac{3}{2}}}+\|\tilde{G}(\tau)\|_{\tilde{B}_{2,1}^{s-1,s-\frac{1}{2}}}\big)\,d\tau\right).$$

Combining the last inequality with (4.13), we complete the proof of Proposition 3.2 as long as we change the functions $(\tilde{h},\tilde{c},\tilde{F},\tilde{G})$ back into the original ones (h,c,F,G).

5. A global existence and uniqueness result

This section is devoted to the proof of Theorem 1.1. The principle of the proof is a very classical one. We shall use the classical Friedrichs' regularization method, which was used in [1,2,15] for examples, to construct the approximate solutions $(h^n, \mathbf{u}^n)$ of (3.4).

5.1. Building of the sequence $(h^n, \mathbf{u}^n)_{n\in\mathbb{N}}$

Let us define the sequence of operators $(\mathscr{J}_n)_{n\in\mathbb{N}}$ by

$$\mathscr{J}_n f := \mathscr{F}^{-1}\mathbf{1}_{B(\frac{1}{n},n)}(\xi)\mathscr{F} f.$$

We consider the approximate system:

$$\begin{cases} h^n_t + \mathscr{J}_n\Lambda^{-1}\big(\mathscr{J}_n\mathbf{u}^n\cdot\nabla\Lambda\mathscr{J}_n h^n\big) + \bar\rho\mathscr{J}_n \mathrm{c}^n = F^n,\\ \mathrm{c}^n_t + \mathscr{J}_n\big(\mathscr{J}_n\mathbf{u}^n\cdot\nabla\mathscr{J}_n\mathrm{c}^n\big) - \bar\rho^{-1}(2\mu+\lambda)\Delta\mathscr{J}_n\mathrm{c}^n - \Lambda^2\mathscr{J}_n h^n - \mathscr{J}_n h^n = G^n,\\ \mathbf{I}^n_t - \bar\rho^{-1}\mu\Delta\mathscr{J}_n\mathbf{I}^n = H^n,\\ \mathbf{u}^n = -\Lambda^{-1}\nabla\mathrm{c}^n - \Lambda^{-1}\operatorname{div}\mathbf{I}^n,\\ \big(h^n,\mathrm{c}^n,\mathbf{I}^n\big)(0) = \big(h_n,\Lambda^{-1}\operatorname{div}\mathbf{u}_n,\Lambda^{-1}\operatorname{curl}\mathbf{u}_n\big), \end{cases} \tag{5.1}$$

where

$$\begin{aligned} &h_n = \mathscr{J}_n(\rho_0-\bar\rho), \qquad \mathbf{u}_n = \mathscr{J}_n\mathbf{u}_0,\\ &F^n = -\mathscr{J}_n\Lambda^{-1}\big(\Lambda\mathscr{J}_n h^n \operatorname{div}\mathscr{J}_n\mathbf{u}^n\big),\\ &G^n = \mathscr{J}_n\big(\mathscr{J}_n\mathbf{u}^n\cdot\nabla\mathscr{J}_n\mathrm{c}^n\big) - \mathscr{J}_n\Lambda^{-1}\operatorname{div} J^n,\\ &H^n = -\mathscr{J}_n\Lambda^{-1}\operatorname{curl} J^n,\\ &J^n = \mathscr{J}_n\big(\mathscr{J}_n\mathbf{u}^n\cdot\nabla\mathscr{J}_n\mathbf{u}^n\big) + \frac{\Lambda\mathscr{J}_n h^n}{\bar\rho\zeta(\Lambda\mathscr{J}_n h^n+\bar\rho)}\big(\mu\Delta\mathscr{J}_n\mathbf{u}^n + (\mu+\lambda)\nabla\operatorname{div}\mathscr{J}_n\mathbf{u}^n\big), \end{aligned}$$

where ζ is a smooth function satisfying

$$\zeta(s) = \begin{cases} \bar\rho/4, & |s|\leqslant\bar\rho/4,\\ s, & \bar\rho/2\leqslant|s|\leqslant 3\bar\rho/2,\\ 7\bar\rho/4, & |s|\geqslant 7\bar\rho/4,\\ \text{smooth}, & \text{otherwise}. \end{cases}$$

We want to show that (5.1) is only an ordinary differential equation in $L^2\times L^2\times L^2$. We can observe easily that all the source term in (5.1) turn out to be continuous in $L^2\times L^2\times L^2$. For example, we consider the term $\mathscr{J}_n\Lambda^{-1}\operatorname{div}\frac{\Lambda\mathscr{J}_n h^n\Delta\mathscr{J}_n\mathbf{u}^n}{\zeta(\Lambda\mathscr{J}_n h^n+\bar\rho)}$. By Plancherel's theorem, Hausdorff–Young's inequality and Hölder's inequality, we have

$$\begin{aligned} \left\|\mathscr{J}_n\Lambda^{-1}\operatorname{div}\frac{\Lambda\mathscr{J}_n h^n\Delta\mathscr{J}_n\mathbf{u}^n}{\zeta(\Lambda\mathscr{J}_n h^n+\bar\rho)}\right\|_{L^2} &= \left\|\mathbf{1}_{B(\frac{1}{n},n)}|\xi|^{-1}\xi\cdot\mathscr{F}\frac{\Lambda\mathscr{J}_n h^n\Delta\mathscr{J}_n\mathbf{u}^n}{\zeta(\Lambda\mathscr{J}_n h^n+\bar\rho)}\right\|_{L^2}\\ &\leqslant \left\|\frac{\Lambda\mathscr{J}_n h^n\Delta\mathscr{J}_n\mathbf{u}^n}{\zeta(\Lambda\mathscr{J}_n h^n+\bar\rho)}\right\|_{L^2} \leqslant \left\|\Lambda\mathscr{J}_n h^n\Delta\mathscr{J}_n\mathbf{u}^n\right\|_{L^2}\left\|\frac{1}{\zeta(\Lambda\mathscr{J}_n h^n+\bar\rho)}\right\|_{L^\infty} \end{aligned}$$

$$\leqslant \frac{4}{\bar\rho}\left\|\Lambda \mathscr{J}_n h^n\right\|_{L^\infty}\left\|\Delta \mathscr{J}_n \mathbf{u}^n\right\|_{L^2} \leqslant \frac{4n^2}{\bar\rho}\left\||\xi|\mathbf{1}_{B(\frac{1}{n},n)}\mathscr{F}h^n\right\|_{L^1}\left\|\mathbf{u}^n\right\|_{L^2}$$
$$\lesssim \frac{4n^{\frac{N}{2}+3}}{\bar\rho}\left\|h^n\right\|_{L^2}\left\|\mathbf{u}^n\right\|_{L^2}.$$

Thus, the usual Cauchy–Lipschitz theorem implies the existence of a strictly positive maximal time T_n such that a unique solution exists which is continuous in time with value in $L^2\times L^2\times L^2$. However, as $\mathscr{J}_n^2=\mathscr{J}_n$, we claim that $\mathscr{J}_n(h^n,\mathrm{c}^n,\mathbf{l}^n)$ is also a solution, so uniqueness implies that $\mathscr{J}_n(h^n,\mathrm{c}^n,\mathbf{l}^n)=(h^n,\mathrm{c}^n,\mathbf{l}^n)$. So $(h^n,\mathrm{c}^n,\mathbf{l}^n)$ is also a solution of the following system:

$$\begin{cases} h_t^n+\mathscr{J}_n\Lambda^{-1}\big(\mathbf{u}^n\cdot\nabla\Lambda h^n\big)+\bar\rho\mathrm{c}^n=F_1^n,\\ \mathrm{c}_t^n+\mathscr{J}_n\big(\mathbf{u}^n\cdot\nabla\mathrm{c}^n\big)-\bar\rho^{-1}(2\mu+\lambda)\Delta\mathrm{c}^n-\Lambda^2h^n-h^n=G_1^n,\\ \mathbf{l}_t^n-\bar\rho^{-1}\mu\Delta\mathbf{l}^n=H_1^n,\\ \mathbf{u}^n=-\Lambda^{-1}\nabla\mathrm{c}^n-\Lambda^{-1}\operatorname{div}\mathbf{l}^n,\\ \big(h^n,\mathrm{c}^n,\mathbf{l}^n\big)(0)=\big(h_n,\Lambda^{-1}\operatorname{div}\mathbf{u}_n,\Lambda^{-1}\operatorname{curl}\mathbf{u}_n\big),\end{cases}\tag{5.2}$$

where

$$F_1^n=-\mathscr{J}_n\Lambda^{-1}\big(\Lambda h^n\operatorname{div}\mathbf{u}^n\big),$$
$$G_1^n=\mathscr{J}_n\big(\mathbf{u}^n\cdot\nabla\mathrm{c}^n\big)-\mathscr{J}_n\Lambda^{-1}\operatorname{div}J^n,$$
$$H_1^n=-\mathscr{J}_n\Lambda^{-1}\operatorname{curl}J^n,$$
$$J_1^n=\mathscr{J}_n\big(\mathbf{u}^n\cdot\nabla\mathbf{u}^n\big)+\frac{\Lambda h^n}{\bar\rho\zeta(\Lambda h^n+\bar\rho)}\big(\mu\Delta\mathbf{u}^n+(\mu+\lambda)\nabla\operatorname{div}\mathbf{u}^n\big).$$

The system (5.2) appears to be an ordinary differential equation in the space

$$L_n^2:=\left\{a\in L^2\big(\mathbb{R}^N\big):\ \operatorname{supp}\mathscr{F}a\subset B\left(\frac{1}{n},n\right)\right\}.$$

Due to the Cauchy–Lipschitz theorem again, a unique maximal solution exists on an interval $[0,T_n^*)$ which is continuous in time with value in $L_n^2\times L_n^2\times L_n^2$.

5.2. Uniform bounds

In this part, we prove uniform estimates independent of $T<T_n^*$ in $E^{\frac{N}{2}}$ for $(h^n,\mathbf{u}^n)$. We shall show that $T_n^*=+\infty$ by the Cauchy–Lipschitz theorem. Define

$$E(0):=\left\|\Lambda^{-1}(\rho_0-\bar\rho)\right\|_{\tilde B_{2,1}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}+1}}+\|\mathbf{u}_0\|_{\tilde B_{2,1}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-1}},$$
$$E(h,\mathbf{u},t):=\left\|(h,\mathbf{u})\right\|_{E_t^{\frac{N}{2}}},$$
$$\tilde T_n:=\sup\big\{t\in\big[0,T_n^*\big):\ E\big(h^n,\mathbf{u}^n,t\big)\leqslant A\tilde C E(0)\big\},$$

where $\tilde C$ corresponds to the constant in Proposition 3.2 and $A>\max(2,\tilde C^{-1})$ is a constant. Thus, by the continuity we have $\tilde T_n>0$.

We are going to prove that $\tilde T_n=T_n^*$ for all $n\in\mathbb{N}$ and we will conclude that $T_n^*=+\infty$ for any $n\in\mathbb{N}$.

According to Proposition 3.2 and Lemma 3.1, and to the definition of $(h_n, \mathbf{u}_n)$, the following inequality holds

$$\begin{aligned}\left\|\left(h^n, \mathbf{u}^n\right)\right\|_{E_T^{\frac{N}{2}}} \leqslant \tilde{C} e^{\tilde{C}\|\mathbf{u}^n\|_{L_T^1(\dot{B}_{2,1}^{\frac{N}{2}+1})}} \Big(&\left\|\Lambda^{-1}(\rho_0-\bar{\rho})\right\|_{\tilde{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}, \frac{N}{2}+1}} + \|\mathbf{u}_0\|_{\tilde{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}, \frac{N}{2}-1}} + \left\|F_1^n\right\|_{L_T^1(\tilde{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}, \frac{N}{2}+1})} \\ &+ \left\|\mathbf{u}^n \cdot \nabla \mathrm{c}^n\right\|_{L_T^1(\tilde{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}, \frac{N}{2}-1})} + \left\|J_1^n\right\|_{L_T^1(\tilde{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}, \frac{N}{2}-1})}\Big).\end{aligned}$$

Therefore, it is only a matter of proving appropriate estimates for F_1^n, J_1^n and the convection term. The estimate of F_1^n is straightforward. From Lemma 2.6, we have

$$\begin{aligned}\left\|F_1^n\right\|_{L_T^1(\tilde{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}, \frac{N}{2}+1})} &= \left\|\Lambda h^n \operatorname{div} \mathbf{u}^n\right\|_{L_T^1(\tilde{B}_{2,1}^{\frac{N}{2}-\frac{5}{2}, \frac{N}{2}})} \\ &\leqslant C\left\|\Lambda h^n\right\|_{L_T^\infty(\tilde{B}_{2,1}^{\frac{N}{2}-2, \frac{N}{2}})} \left\|\operatorname{div} \mathbf{u}^n\right\|_{L_T^1(\tilde{B}_{2,1}^{\frac{N}{2}-\frac{1}{2}, \frac{N}{2}})} \\ &\leqslant C\left\|h^n\right\|_{L_T^\infty(\tilde{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}, \frac{N}{2}+1})} \left\|\mathbf{u}^n\right\|_{L_T^1(\tilde{B}_{2,1}^{\frac{N}{2}+\frac{1}{2}, \frac{N}{2}+1})} \\ &\leqslant C E^2\left(h^n, \mathbf{u}^n, T\right). \qquad (5.3)\end{aligned}$$

With the help of Lemma 2.6 and interpolation arguments, we have

$$\begin{aligned}\left\|\mathbf{u}^n \cdot \nabla \mathrm{c}^n\right\|_{L_T^1(\tilde{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}, \frac{N}{2}-1})} &\leqslant C\left\|\mathbf{u}^n\right\|_{L_T^\infty(\tilde{B}_{2,1}^{\frac{N}{2}-1, \frac{N}{2}-1})} \left\|\nabla \mathrm{c}^n\right\|_{L_T^1(\tilde{B}_{2,1}^{\frac{N}{2}-\frac{1}{2}, \frac{N}{2}})} \\ &\leqslant C\left\|\mathbf{u}^n\right\|_{L_T^\infty(\tilde{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}, \frac{N}{2}-1})} \left\|\mathbf{u}^n\right\|_{L_T^1(\tilde{B}_{2,1}^{\frac{N}{2}+\frac{1}{2}, \frac{N}{2}+1})} \\ &\leqslant C E^2\left(h^n, \mathbf{u}^n, T\right). \qquad (5.4)\end{aligned}$$

In the same way, we can get

$$\left\|\mathbf{u}^n \cdot \nabla \mathbf{u}^n\right\|_{L_T^1(\tilde{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}, \frac{N}{2}-1})} \leqslant C E^2\left(h^n, \mathbf{u}^n, T\right). \qquad (5.5)$$

To estimate other terms of J_1^n, we make the following assumption on $E(0)$:

$$2 C_1 A \tilde{C} E(0) \leqslant \bar{\rho}, \qquad (5.6)$$

where C_1 is the continuity modulus of the embedding relation $\dot{B}_{2,1}^{\frac{N}{2}}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$. If $T < \tilde{T}_n$, it implies

$$\begin{aligned}\left\|\Lambda h^n\right\|_{L^\infty([0,T]\times\mathbb{R}^N)} \leqslant C_1\left\|h^n\right\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{N}{2}+1})} &\leqslant C_1\left\|h^n\right\|_{L_T^\infty(\tilde{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}, \frac{N}{2}+1})} \\ &\leqslant C_1 A \tilde{C} E(0) \leqslant \frac{1}{2}\bar{\rho}, \qquad (5.7)\end{aligned}$$

which yields

$$\Lambda h^n + \bar{\rho} \in \left[\frac{1}{2}\bar{\rho}, \frac{3}{2}\bar{\rho}\right] \quad \text{and} \quad \zeta\left(\Lambda h^n + \bar{\rho}\right) = \Lambda h^n + \bar{\rho}.$$

From Lemmas 2.6 and 2.3, we obtain

$$\begin{aligned}
&\left\| \frac{\Lambda h^n}{\bar{\rho}(\Lambda h^n + \bar{\rho})}\big(\mu\Delta\mathbf{u}^n + (\mu+\lambda)\nabla\operatorname{div}\mathbf{u}^n\big)\right\|_{L^1_T(\tilde{B}_{2,1}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-1})} \\
&\leqslant C\left\|\frac{\Lambda h^n}{\Lambda h^n+\bar{\rho}}\right\|_{L^\infty_T(\dot{B}_{2,1}^{\frac{N}{2}})}\|\mathbf{u}^n\|_{L^1_T(\tilde{B}_{2,1}^{\frac{N}{2}+\frac{1}{2},\frac{N}{2}+1})} \\
&\leqslant C\|\Lambda h^n\|_{L^\infty_T(\dot{B}_{2,1}^{\frac{N}{2}})}\|\mathbf{u}^n\|_{L^1_T(\tilde{B}_{2,1}^{\frac{N}{2}+\frac{1}{2},\frac{N}{2}+1})} \\
&\leqslant C\|h^n\|_{L^\infty_T(\tilde{B}_{2,1}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}+1})}\|\mathbf{u}^n\|_{L^1_T(\tilde{B}_{2,1}^{\frac{N}{2}+\frac{1}{2},\frac{N}{2}+1})} \\
&\leqslant CE^2\big(h^n,\mathbf{u}^n,T\big).
\end{aligned} \tag{5.8}$$

Thus, we get

$$\big\|\big(h^n,\mathbf{u}^n\big)\big\|_{E_T^{\frac{N}{2}}} \leqslant \tilde{C}e^{A\tilde{C}^2E(0)}\big[1+CA^2\tilde{C}^2E(0)\big]E(0).$$

So we can choose $E(0)$ so small that

$$1+CA^2\tilde{C}^2E(0)\leqslant\frac{A^2}{A+2},\qquad e^{A\tilde{C}E(0)}\leqslant\frac{A+1}{A},\qquad 2C_1A\tilde{C}E(0)\leqslant\bar{\rho}, \tag{5.9}$$

which yields $\|(h^n,\mathbf{u}^n)\|_{E^1_T}\leqslant\frac{A+1}{A+2}A\tilde{C}E(0)$ for any $T<\tilde{T}_n$. It follows that $\tilde{T}_n=T_n^*$. In fact, if $\tilde{T}_n<T_n^*$, we have seen that $E(h^n,\mathbf{u}^n,\tilde{T}_n)\leqslant\frac{A+1}{A+2}A\tilde{C}E(0)$. So by continuity, for a sufficiently small constant $\sigma>0$ we can obtain $E(h^n,\mathbf{u}^n,\tilde{T}_n+\sigma)\leqslant A\tilde{C}E(0)$. This yields a contradiction with the definition of $\tilde{T}_n$.

Now, if $\tilde{T}_n=T_n^*<\infty$, we have obtained $F(h^n,\mathbf{u}^n,T_n^*)\leqslant A\tilde{C}E(0)$. As $\|h^n\|_{L_{T_n^*}(\tilde{B}_{2,1}^{0,1+\varepsilon})}<\infty$ and $\|\mathbf{u}^n\|_{L_{T_n^*}(\tilde{B}_{2,1}^{0,\varepsilon})}<\infty$, it implies that $\|h^n\|_{L_{T_n^*}(L_n^2)}<\infty$ and $\|\mathbf{u}^n\|_{L_{T_n^*}(L_n^2)}<\infty$. Thus, we may continue the solution beyond T_n^* by the Cauchy–Lipschitz theorem. This contradicts the definition of T_n^*. Therefore, the approximate solution $(h^n,\mathbf{u}^n)_{n\in\mathbb{N}}$ is global in time.

5.3. Existence of a solution

In this part, we shall show that, up to an extraction, the sequence $(h^n,\mathbf{u}^n)_{n\in\mathbb{N}}$ converges in $\mathscr{D}'(\mathbb{R}^+\times\mathbb{R}^N)$ to a solution $(h,\mathbf{u})$ of (3.4) which has the desired regularity properties. The proof lies on compactness arguments. To start with, we show that the time first derivative of $(h^n,\mathbf{u}^n)$ is uniformly bounded in appropriate spaces. This enables us to apply Ascoli's theorem and get the existence of a limit $(h,\mathbf{u})$ for a subsequence. Now, the uniform bounds of the previous part provides us with additional regularity and convergence properties so that we may pass to the limit in the system.

It is convenient to split $(h^n,\mathbf{u}^n)$ into the solution of a linear system with initial data $(h_n,\mathbf{u}_n)$, and the discrepancy to that solution. More precisely, we denote by $(h_L^n,\mathbf{u}_L^n)$ the solution to the linear system

$$\begin{cases}
\partial_t h_L^n+\bar{\rho}\Lambda^{-1}\operatorname{div}\mathbf{u}_L^n=0,\\
\partial_t\mathbf{u}_L^n-\bar{\rho}^{-1}\mu\Delta\mathbf{u}_L^n-\bar{\rho}^{-1}(\mu+\lambda)\nabla\operatorname{div}\mathbf{u}^n+\nabla\Lambda h_L^n+\nabla\Lambda^{-1}h_L^n=0,\\
\big(h_L^n,\mathbf{u}_L^n\big)_{t=0}=(h_n,\mathbf{u}_n),
\end{cases} \tag{5.10}$$

and $(\bar{h}^n,\bar{\mathbf{u}}^n)=(h^n-h_L^n,\mathbf{u}^n-\mathbf{u}_L^n)$.

Obviously, the definition of $(h_n, \mathbf{u}_n)$ entails

$$h_n \to \Lambda^{-1}(\rho_0 - \bar{\rho}) \quad \text{in } \tilde{B}_{2,1}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}+1}, \qquad \mathbf{u}_n \to \mathbf{u}_0 \quad \text{in } \tilde{B}_{2,1}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-1} \text{ as } n \to \infty.$$

The Lemma 3.1 and Proposition 3.2 insure us that

$$\left(h_L^n, \mathbf{u}_L^n\right) \to (h_L, \mathbf{u}_L) \quad \text{in } E^{\frac{N}{2}}, \tag{5.11}$$

where $(h_L, \mathbf{u}_L)$ is the solution of the linear system

$$\begin{cases} \partial_t h_L + \bar{\rho}\Lambda^{-1} \operatorname{div} \mathbf{u}_L = 0, \\ \partial_t \mathbf{u}_L - \bar{\rho}^{-1}\mu\Delta\mathbf{u}_L - \bar{\rho}^{-1}(\mu+\lambda)\nabla \operatorname{div} \mathbf{u} + \nabla\Lambda h_L + \nabla\Lambda^{-1} h_L = 0, \\ (h_L, \mathbf{u}_L)_{t=0} = \left(\Lambda^{-1}(\rho_0 - \bar{\rho}), \mathbf{u}_0\right). \end{cases} \tag{5.12}$$

Now, we have to prove the convergence of $(\bar{h}^n, \bar{\mathbf{u}}^n)$. This is of course a trifle more difficult and requires compactness results. Let us first state the following lemma.

Lemma 5.1. $((\bar{h}^n, \bar{\mathbf{u}}^n))_{n\in\mathbb{N}}$ *is uniformly bounded in*

$$\mathcal{C}^{\frac{1}{2}}\left(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}}\right) \times \left(\mathcal{C}^{\frac{1}{4}}\left(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}}\right)\right)^N.$$

Proof. Throughout the proof, we will note u.b. for uniformly bounded. We first prove that $\partial_t \bar{h}^n$ is u.b. in $(L^2 + L^\infty)(\mathbb{R}^+, \dot{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}})$, which yields the desired result for $\bar{h}$. Let us observe that $\bar{h}^n$ verifies the following equation

$$\partial_t \bar{h}^n = -\mathscr{J}_n \Lambda^{-1}\left(\Lambda h^n \operatorname{div} \mathbf{u}^n\right) - \mathscr{J}_n \Lambda^{-1}\left(\mathbf{u}^n \cdot \nabla \Lambda h^n\right) - \bar{\rho}\Lambda^{-1} \operatorname{div} \mathbf{u}^n + \bar{\rho}\Lambda^{-1} \operatorname{div} \mathbf{u}_L^n.$$

According to the previous part, $(h^n)_{n\in\mathbb{N}}$ is u.b. in $L^\infty(\dot{B}_{2,1}^{\frac{N}{2}-\frac{1}{2}})$ and $(\mathbf{u}^n)_{n\in\mathbb{N}}$ is u.b. in $L^2(\dot{B}_{2,1}^{\frac{N}{2}})$ in view of interpolation arguments. Thus, $\mathscr{J}_n \Lambda^{-1}(\Lambda h^n \operatorname{div} \mathbf{u}^n)$, $\mathscr{J}_n \Lambda^{-1}(\mathbf{u}^n \cdot \nabla \Lambda h^n)$, $\bar{\rho}\Lambda^{-1} \operatorname{div} \mathbf{u}^n$ is u.b. in $L^2(\dot{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}})$. The definition of $\mathbf{u}_L^n$ obviously provides us with uniform bounds for $\Lambda^{-1} \operatorname{div} \mathbf{u}_L^n$ in $L^\infty(\dot{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}})$, so we can conclude that $\partial_t \bar{h}^n$ is u.b. in $(L^2 + L^\infty)(\dot{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}})$.

Denote $\mathrm{c}_L^n = \Lambda^{-1} \operatorname{div} \mathbf{u}_L^n$, $\bar{\mathrm{c}}^n = \Lambda^{-1} \operatorname{div} \bar{\mathbf{u}}^n$, $\mathbf{I}_L^n = \Lambda^{-1} \operatorname{curl} \mathbf{u}_L^n$ and $\bar{\mathbf{I}}^n = \Lambda^{-1} \operatorname{curl} \bar{\mathbf{u}}^n$. Let us prove now that $\partial_t \bar{\mathrm{c}}^n$ is u.b. in $(L^{\frac{4}{3}} + L^\infty)(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}})$ and that $\partial_t \bar{\mathbf{I}}^n$ is u.b. in $L^{\frac{4}{3}}(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}})$ which give the required result for $\bar{\mathbf{u}}^n$ by using the relation $\mathbf{u}^n = -\Lambda^{-1}\nabla \mathrm{c}^n - \Lambda^{-1} \operatorname{div} \mathbf{I}^n$.

Let us recall that

$$\partial_t \bar{\mathrm{c}}^n = \bar{\rho}^{-1}(2\mu+\lambda)\Delta\left(\mathrm{c}^n - \mathrm{c}_L^n\right) + \Lambda^2\left(h^n - h_L^n\right) + \left(h^n - h_L^n\right) - \mathscr{J}_n \Lambda^{-1} \operatorname{div} J^n,$$

$$\partial_t \bar{\mathbf{I}}^n = \bar{\rho}^{-1}\mu\Delta\left(\mathbf{I}^n - \mathbf{I}_L^n\right) - \mathscr{J}_n \Lambda^{-1} \operatorname{curl} J^n.$$

Results of the previous part and an interpolation argument yield uniform bounds for $\mathbf{u}^n$ and c^n in $L^{\frac{4}{3}}(\dot{B}_{2,1}^{\frac{N}{2}+\frac{1}{2}}) \cap L^2(\dot{B}_{2,1}^{\frac{N}{2}})$. Since h^n is u.b. in $L^\infty(\dot{B}_{2,1}^{\frac{N}{2}+1})$ and c_L^n is u.b. in $L^{\frac{4}{3}}(\dot{B}_{2,1}^{\frac{N}{2}+\frac{1}{2}})$, we easily verify that $\Delta(\mathrm{c}^n - \mathrm{c}_L^n)$ and $\mathscr{J}_n \Lambda^{-1} \operatorname{div} J^n$ are u.b. in $L^{\frac{4}{3}}(\dot{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}})$. Because h^n is u.b. in $L^\infty(\dot{B}_{2,1}^{\frac{N}{2}+\frac{1}{2}})$, $\Lambda^2 h^n$ is u.b. in $L^\infty(\dot{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}})$. We also have $\Lambda^2 h_L^n$ u.b. in $L^\infty(\dot{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}})$. In addition, h^n and h_L^n are u.b. in $L^\infty(\dot{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}})$. So we finally get $\partial_t \bar{\mathrm{c}}^n$ u.b. in $(L^{\frac{4}{3}} + L^\infty)(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}})$. The case of $\partial_t \bar{\mathbf{l}}^n$ goes along the same lines. As the terms corresponding to $(h^n - h_L^n)$ do not appear, we simply get $\partial_t \bar{\mathbf{l}}^n$ u.b. in $L^{\frac{4}{3}}(\dot{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}})$. $\square$

Now, we can turn to the proof of the existence of a solution and use Ascoli theorem to get strong convergence. We need to localize the spatial space because we have some results of compactness for the local Sobolev spaces. Let $(\chi_p)_{p\in\mathbb{N}}$ be a sequence of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ cut-off functions supported in the ball $B(0, p+1)$ of $\mathbb{R}^N$ and equal to 1 in a neighborhood of $B(0, p)$.

For any $p \in \mathbb{N}$, Lemma 5.1 tells us that $((\chi_p \bar{\rho}^n, \chi_p \bar{\mathbf{u}}^n))_{n\in\mathbb{N}}$ is uniformly equicontinuous in $\mathcal{C}(\mathbb{R}^+; (\dot{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}})^{1+N})$.

Let us observe that the application $f \mapsto \chi_p f$ is compact from $\tilde{B}_{2,1}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}+1}$ into $\dot{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}}$, and from $\tilde{B}_{2,1}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-1}$ into $\dot{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}}$. After we apply Ascoli's theorem to the family $((\chi_p \bar{h}^n, \chi_p \bar{\mathbf{u}}^n))_{n\in\mathbb{N}}$ on the time interval $[0, p]$, we use Cantor's diagonal process. This finally provides us with a distribution $(\bar{h}, \bar{\mathbf{u}})$ belonging to $\mathcal{C}(\mathbb{R}^+; (\dot{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}})^{1+N})$ and a subsequence (which we still denote by $((\bar{\rho}^n, \bar{\mathbf{u}}^n)_{n\in\mathbb{N}})$) such that, for all $p \in \mathbb{N}$, we have

$$\left(\chi_p \bar{h}^n, \chi_p \bar{\mathbf{u}}^n\right) \to (\chi_p \bar{h}, \chi_p \bar{\mathbf{u}}) \quad \text{as } n \to +\infty, \text{ in } \mathcal{C}\left([0, p]; \left(\dot{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}}\right)^{1+N}\right). \tag{5.13}$$

This obviously infers that $(\bar{h}^n, \bar{\mathbf{u}}^n)$ tends to $(\bar{h}, \bar{\mathbf{u}})$ in $\mathscr{D}'(\mathbb{R}^+ \times \mathbb{R}^N)$.

Coming back to the uniform estimates of the previous part, we moreover get that $(\bar{h}, \bar{\mathbf{u}})$ belongs to

$$L^\infty\left(\mathbb{R}^+; \tilde{B}_{2,1}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}+1} \times \left(\tilde{B}_{2,1}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-1}\right)^N\right) \cap L^1\left(\mathbb{R}^+; \left(\tilde{B}_{2,1}^{\frac{N}{2}+\frac{1}{2},\frac{N}{2}+1}\right)^{1+N}\right)$$

and to $\mathcal{C}^{1/2}(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}}) \times (\mathcal{C}^{1/4}(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{N}{2}-\frac{3}{2}}))^N$.

Let us now prove that $(h, \mathbf{u}) := (h_L, \mathbf{u}_L) + (\bar{h}, \bar{\mathbf{u}})$ solves (3.4). We first observe that, according to (5.1),

$$\begin{cases} h_t^n + \mathscr{J}_n \Lambda^{-1}\left(\mathbf{u}^n \cdot \nabla \Lambda h^n\right) + \bar{\rho}\mathrm{c}^n = -\mathscr{J}_n \Lambda^{-1}\left(\Lambda h^n \operatorname{div} \mathbf{u}^n\right), \\ \mathbf{u}_t^n + \mathscr{J}_n\left(\mathbf{u}^n \cdot \nabla \mathbf{u}^n\right) - \bar{\rho}^{-1}\mu\Delta\mathbf{u}^n - \bar{\rho}^{-1}(\mu+\lambda)\nabla \operatorname{div} \mathbf{u}^n + \Lambda\nabla h^n + \Lambda^{-1}\nabla h^n \\ \quad = -\mathscr{J}_n \dfrac{\Lambda h^n}{\bar{\rho}(\Lambda h^n + \bar{\rho})}\left(\mu\Delta\mathbf{u}^n + (\mu+\lambda)\nabla \operatorname{div} \mathbf{u}^n\right). \end{cases} \tag{5.14}$$

The only problem is to pass to the limit in $\mathscr{D}'(\mathbb{R}^+ \times \mathbb{R}^N)$ in nonlinear terms. This can be done by using the convergence results stemming from the uniform estimates and the convergence results (5.11) and (5.12).

As it is just a matter of doing tedious verifications, we show, as an example, the case of the term $\mathscr{J}_n \frac{\Lambda h^n \Delta \mathbf{u}^n}{\bar{\rho}(\Lambda h^n + \bar{\rho})}$. Denote $L(z) = z/(z + \bar{\rho})$. Let $\theta \in \mathcal{C}_0^\infty(\mathbb{R}^+ \times \mathbb{R}^N)$ and $p \in \mathbb{N}$ be such that $\operatorname{supp}\theta \subset [0, p] \times B(0, p)$. We consider the decomposition

$$\begin{aligned} &\mathscr{J}_n \frac{\theta \Lambda h^n \Delta \mathbf{u}^n}{\bar{\rho}(\Lambda h^n + \bar{\rho})} - \frac{\theta \Lambda h \Delta \mathbf{u}}{\bar{\rho}(\Lambda h + \bar{\rho})} \\ &\quad = \bar{\rho}^{-2} \mathscr{J}_n\left[\theta\left(1 - L\left(\Lambda h^n\right)\right)\chi_p \Lambda h^n \chi_p \Delta\left(\mathbf{u}_L^n - \mathbf{u}_L\right) + \theta\left(1 - L\left(\Lambda h^n\right)\right)\chi_p \Lambda h^n \chi_p \Delta\left(\chi_p\left(\bar{\mathbf{u}}^n - \bar{\mathbf{u}}\right)\right)\right. \\ &\qquad \left. + \theta\left(1 - L\left(\Lambda h^n\right)\right)\left(\chi_p \Lambda\left(h^n - h\right)\right)\Delta\mathbf{u} - \theta \Lambda h \chi_p \Delta\mathbf{u}\left(L\left(\chi_p \Lambda h^n\right) - L(\chi_p \Lambda h)\right)\right] \\ &\qquad + (\mathscr{J}_n - I)\frac{\theta \Lambda h \Delta \mathbf{u}}{\bar{\rho}(\Lambda h + \bar{\rho})}. \end{aligned}$$

The last term tends to zero as $n \to +\infty$ due to the property of $\mathscr{J}_n$. As $\theta L(\Lambda h^n)$ and Λh^n are u.b. in $L^\infty(\dot B^{\frac N2}_{2,1})$ and $\mathbf{u}^n_L$ tends to $\mathbf{u}_L$ in $L^{\frac43}(\dot B^{\frac N2-\frac32}_{2,1})$, the first term tends to 0 in $L^{\frac43}(\dot B^{\frac N2-\frac32}_{2,1})$. According to (5.12), $\chi_p(\bar{\mathbf{u}}^n-\bar{\mathbf{u}})$ tends to zero in $L^{\frac43}(\dot B^{\frac N2-\frac32}_{2,1})$ so that the second term tends to 0 in $L^{\frac43}(\dot B^{\frac N2-\frac32}_{2,1})$. Clearly, $\chi_p\Lambda h^n\to\chi_p\Lambda h$ in $L^\infty(\dot B^{\frac N2}_{2,1})$ and $L(\chi_p\Lambda h^n)\to L(\chi_p\Lambda h)$ in $L^\infty(L^\infty\cap\dot B^{\frac N2}_{2,1})$, so that the third and the last terms also tend to 0 in $L^{\frac43}(\dot B^{\frac N2-\frac32}_{2,1})$. The other nonlinear terms can be treated in the same way.

We still have to prove that h is continuous in $\tilde B^{\frac N2-\frac32,\frac N2+1}_{2,1}$ and that $\mathbf{u}$ belongs to $\mathcal{C}(\mathbb{R}^+;\tilde B^{\frac N2-\frac32,\frac N2-\frac12}_{2,1})$. The continuity of $\mathbf{u}$ is straightforward. Indeed, $\mathbf{u}$ satisfies

$$\begin{aligned}\mathbf{u}_t=&-\mathbf{u}\cdot\nabla\mathbf{u}+\bar\rho^{-1}\mu\Delta\mathbf{u}+\bar\rho^{-1}(\mu+\lambda)\nabla\operatorname{div}\mathbf{u}\\&-\Lambda\nabla h-\Lambda^{-1}\nabla h-\frac{\Lambda h}{\bar\rho(\Lambda h+\bar\rho)}\big(\mu\Delta\mathbf{u}+(\mu+\lambda)\nabla\operatorname{div}\mathbf{u}\big)\end{aligned}$$

and the r.h.s. belongs to $(L^1+L^\infty)(\mathbb{R}^+;\tilde B^{\frac N2-\frac32,\frac N2-1}_{2,1})$. We have already got that $h\in\mathcal{C}(\mathbb{R}^+;\dot B^{\frac N2-\frac32}_{2,1})$. Indeed, $h_t\in L^\infty(\mathbb{R}^+;\dot B^{\frac N2-\frac32}_{2,1})$ from the equation

$$h_t=-\Lambda^{-1}\operatorname{div}(\Lambda h\mathbf{u})-\bar\rho\Lambda^{-1}\operatorname{div}\mathbf{u}.$$

Thus, there remains to prove the continuity of h in $\dot B^{\frac N2+1}_{2,1}$.

Let us apply the operator dk to the first equation of (3.4) to get

$$\partial_t\triangle_k\Lambda h=-\triangle_k(\mathbf{u}\cdot\nabla\Lambda h)-\bar\rho\triangle_k\operatorname{div}\mathbf{u}-\triangle_k(\Lambda h\operatorname{div}\mathbf{u}).\tag{5.15}$$

Obviously, for fixed k the r.h.s. belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^+;L^2)$ so that each $\triangle_k\Lambda h$ is continuous in time with values in L^2.

Now, we apply an energy method to (5.13) to obtain, with the help of Lemma 2.7, that

$$\frac12\frac{d}{dt}\|\triangle_k\Lambda h\|^2_{L^2}\leqslant C\|\triangle_k\Lambda h\|_{L^2}\big(\gamma_k2^{-k\frac N2}\|\Lambda h\|_{\dot B^{\frac N2}_{2,1}}\|\mathbf{u}\|_{\dot B^{\frac N2+1}_{2,1}}+\|\triangle_k\operatorname{div}\mathbf{u}\|_{L^2}+\big\|\triangle_k(\Lambda h\operatorname{div}\mathbf{u})\big\|_{L^2}\big),$$

where $\sum_{k\in\mathbb{Z}}\gamma_k\leqslant1$. Integrating in time and multiplying $2^{k\frac N2}$, we get

$$\begin{aligned}2^{k(\frac N2+1)}\big\|\triangle_kh(t)\big\|_{L^2}\leqslant{}&2^{k(\frac N2+1)}\big\|\triangle_k\Lambda^{-1}(\rho_0-\bar\rho)\big\|_{L^2}+C\int_0^t\big(\gamma_k\big\|h(\tau)\big\|_{\dot B^{\frac N2+1}_{2,1}}\big\|\mathbf{u}(\tau)\big\|_{\dot B^{\frac N2+1}_{2,1}}\\&+2^{k(\frac N2+1)}\big\|\triangle_k\mathbf{u}(\tau)\big\|_{L^2}+2^{k\frac N2}\big\|\triangle_k(\Lambda h\operatorname{div}\mathbf{u})(\tau)\big\|_{L^2}\big)\,d\tau.\end{aligned}$$

Since $h\in L^\infty(\dot B^{\frac N2+1}_{2,1})$, $\mathbf{u}\in L^1(\dot B^{\frac N2+1}_{2,1})$ and $\Lambda h\operatorname{div}\mathbf{u}\in L^1(\dot B^{\frac N2}_{2,1})$, we can get

$$\begin{aligned}\sum_{k\in\mathbb{Z}}2^{k(\frac N2+1)}\big\|\triangle_kh(t)\big\|_{L^2}\lesssim{}&\|\rho_0-\bar\rho\|_{\dot B^{\frac N2}_{2,1}}+\big(1+\|h\|_{L^\infty(\dot B^{\frac N2+1}_{2,1})}\big)\|\mathbf{u}\|_{L^1(\dot B^{\frac N2+1}_{2,1})}\\&+\|\Lambda h\operatorname{div}\mathbf{u}\|_{L^1(\dot B^{\frac N2}_{2,1})}<\infty.\end{aligned}$$

Thus, $\sum_{|k|\leqslant N}\triangle_kh$ converges uniformly in $L^\infty(\mathbb{R}^+;\dot B^{\frac N2+1}_{2,1})$ and we can conclude that $h\in\mathcal{C}(\mathbb{R}^+;\dot B^{\frac N2+1}_{2,1})$.

5.4. Uniqueness

Let $(h_1,\mathbf{u}_1)$ and $(h_2,\mathbf{u}_2)$ be solutions of

$$\begin{cases} h_t+\Lambda^{-1}(\mathbf{u}\cdot\nabla\Lambda h)+\bar\rho\Lambda^{-1}\operatorname{div}\mathbf{u}=-\Lambda^{-1}(\Lambda h\operatorname{div}\mathbf{u}),\\ \mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}-\bar\rho^{-1}\mu\Delta\mathbf{u}-\bar\rho^{-1}(\mu+\lambda)\nabla\operatorname{div}\mathbf{u}+\Lambda\nabla h+\Lambda^{-1}\nabla h\\ \quad=-\dfrac{\Lambda h}{\bar\rho(\Lambda h+\bar\rho)}\big(\mu\Delta\mathbf{u}+(\mu+\lambda)\nabla\operatorname{div}\mathbf{u}\big)\end{cases}\tag{5.16}$$

in $E_T^{\frac{N}{2}}$ with the same data $(\Lambda^{-1}(\rho_0-\bar\rho),\mathbf{u}_0)$ constructed in the previous parts on the time interval $[0,T]$. Denote $(\delta h,\delta\mathbf{u})=(h_2-h_1,\mathbf{u}_2-\mathbf{u}_1)$. From (5.16), we can get

$$\begin{cases}\partial_t\delta h+\Lambda^{-1}(\mathbf{u}_2\cdot\nabla\Lambda h_2)+\bar\rho\Lambda^{-1}\operatorname{div}\delta\mathbf{u}=F_1,\\ \partial_t\delta\mathbf{u}+\mathbf{u}_2\cdot\nabla\delta\mathbf{u}+\bar\rho^{-1}\mu\Delta\delta\mathbf{u}-\bar\rho^{-1}(\mu+\lambda)\nabla\operatorname{div}\delta\mathbf{u}+\Lambda\nabla\delta h+\Lambda^{-1}\nabla\delta h=F_2,\\ (\delta h,\delta\mathbf{u})=(0,\mathbf{0}),\end{cases}\tag{5.17}$$

where

$$\begin{aligned}F_1&=-\Lambda^{-1}(\delta\mathbf{u}\cdot\nabla h_1)-\Lambda^{-1}(\Lambda\delta h\operatorname{div}\mathbf{u}_2)-\Lambda^{-1}(\Lambda h_1\operatorname{div}\delta\mathbf{u}),\\ F_2&=-\delta\mathbf{u}\cdot\nabla\mathbf{u}_1-\frac{\Lambda h_1}{\bar\rho(\Lambda h_2+\bar\rho)}\big(\mu\Delta\delta\mathbf{u}+(\mu+\lambda)\nabla\operatorname{div}\delta\mathbf{u}\big)\\ &\quad+\left(\frac{1}{\Lambda h_2+\bar\rho}-\frac{1}{\bar\rho}-\frac{1}{\Lambda h_1+\bar\rho}+\frac{1}{\bar\rho}\right)\big(\mu\Delta\mathbf{u}_1+(\mu+\lambda)\nabla\operatorname{div}\mathbf{u}_1\big).\end{aligned}$$

Similar to (3.1), we can get

$$\big\|(\delta h,\delta\mathbf{u})\big\|_{E_T^{\frac{N}{2}}}\leqslant Ce^{C\|\mathbf{u}_2\|_{L_T^1(\dot B_{2,1}^{\frac{N}{2}+1})}}\big(\|F_1\|_{L_T^1(\tilde B_{2,1}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}+1})}+\|F_2\|_{L_T^1(\tilde B_{2,1}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-1})}\big).$$

Noticing that

$$\begin{aligned}h_1,h_2&\in L_T^\infty\big(\tilde B_{2,1}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}+1}\big)\cap L_T^1\big(\tilde B_{2,1}^{\frac{N}{2}+\frac{1}{2},\frac{N}{2}+1}\big),\\ \mathbf{u}_1,\mathbf{u}_2&\in L_T^\infty\big(\tilde B_{2,1}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-1}\big)\cap L_T^1\big(\tilde B_{2,1}^{\frac{N}{2}+\frac{1}{2},\frac{N}{2}+1}\big),\end{aligned}$$

and

$$\|h_1\|_{L^\infty([0,T]\times\mathbb{R}^N)}\leqslant\frac{1}{2}\bar\rho,\qquad\|h_2\|_{L^\infty([0,T]\times\mathbb{R}^N)}\leqslant\frac{1}{2}\bar\rho,$$

by the construction of solutions, we have with the help of interpolation arguments

$$\begin{aligned}\|F_1\|_{L_T^1(\tilde B_{2,1}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}+1})}\lesssim{}&\|h_1\|_{L_T^2(\tilde B_{2,1}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+1})}\|\delta\mathbf{u}\|_{L_T^2(\tilde B_{2,1}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}})}+\|\delta h\|_{L_T^\infty(\tilde B_{2,1}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+1})}\|\mathbf{u}_2\|_{L_T^1(\tilde B_{2,1}^{\frac{N}{2}+\frac{1}{2},\frac{N}{2}+1})}\\ &+\|h_1\|_{L_T^\infty(\tilde B_{2,1}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+1})}\|\delta\mathbf{u}\|_{L_T^1(\tilde B_{2,1}^{\frac{N}{2}+\frac{1}{2},\frac{N}{2}+1})}\end{aligned}$$

and

$$
\begin{aligned}
&\|F_2\|_{L^1_T(\tilde{B}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-1}_{2,1})} \\
&\lesssim \|\delta\mathbf{u}\|_{L^\infty_T(\tilde{B}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-1}_{2,1})}\|\mathbf{u}_1\|_{L^1_T(\dot{B}^{\frac{N}{2}+1}_{2,1})} + \big(1+\|h_2\|_{L^\infty_T(\dot{B}^{\frac{N}{2}+1}_{2,1})}\big)\cdot\|h_1\|_{L^\infty_T(\dot{B}^{\frac{N}{2}+1}_{2,1})}\|\delta\mathbf{u}\|_{L^1_T(\tilde{B}^{\frac{N}{2}+\frac{1}{2},\frac{N}{2}+1}_{2,1})} \\
&\quad + \big(\|h_1\|_{L^\infty_T(\dot{B}^{\frac{N}{2}+1}_{2,1})} + \|h_2\|_{L^\infty_T(\dot{B}^{\frac{N}{2}+1}_{2,1})}\big)\|\delta h\|_{L^\infty_T(\dot{B}^{\frac{N}{2}+1}_{2,1})}\|\mathbf{u}_1\|_{L^1_T(\tilde{B}^{\frac{N}{2}+\frac{1}{2},\frac{N}{2}+1}_{2,1})}.
\end{aligned}
$$

Thus, we obtain

$$
\big\|(\delta h,\delta\mathbf{u})\big\|_{E^{\frac{N}{2}}_T} \leqslant Ce^{C\|\mathbf{u}_2\|_{L^1_T(\dot{B}^{\frac{N}{2}+1}_{2,1})}}\big\{\big(1+\|h_2\|_{L^\infty_T(\dot{B}^{\frac{N}{2}+1}_{2,1})}\big)\|h_1\|_{L^\infty_T(\dot{B}^{\frac{N}{2}+1}_{2,1})} + Z(T)\big\}\big\|(\delta h,\delta\mathbf{u})\big\|_{E^{\frac{N}{2}}_T},
$$

where $\limsup_{T\to 0^+} Z(T) = 0$.

Supposing that $2C(1+\bar{\rho}C_1^{-1})A\tilde{C}E(0) < \frac{1}{4}$ besides the conditions in (5.9), and taking $T>0$ small enough such that $C\|\mathbf{u}_2\|_{L^1_T(\dot{B}^{\frac{N}{2}+1}_{2,1})} \leqslant \ln 2$ and $Z(T) < \frac{1}{2}$, we obtain $\|(\delta h,\delta\mathbf{u})\|_{E^{\frac{N}{2}}_T} \equiv 0$. Hence, $(h_1,\mathbf{u}_1) \equiv (h_2,\mathbf{u}_2)$ on $[0,T]$.

Let T_m (supposedly finite) be the largest time such that the two solutions coincide on $[0,T_m]$. If we denote

$$
\big(\tilde{h}_i(t),\tilde{\mathbf{u}}_i(t)\big) := \big(h_i(t+T_m),\mathbf{u}_i(t+T_m)\big), \quad i=1,2,
$$

we can use the above arguments and the fact that

$$
\|\tilde{h}_i\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^N)} \leqslant \frac{1}{2}\bar{\rho} \quad \text{and} \quad \|\tilde{h}_i\|_{L^\infty(\mathbb{R}^+;\tilde{B}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}+1}_{2,1})} \leqslant A\tilde{C}E(0)
$$

to prove that $(\tilde{h}_1,\tilde{\mathbf{u}}_1) = (\tilde{h}_2,\tilde{\mathbf{u}}_2)$ on the interval $[0,T_m]$ with the same T_m as in the previous. Therefore, we complete the proofs.

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References

[1] J.-Y. Chemin, Localization in Fourier space and Navier–Stokes system, in: Phase Space Analysis of Partial Differential Equations. Proceedings 2004, in: CRM Series, Ed. Norm., Pisa, 2004, pp. 53–136.

[2] J.-Y. Chemin, P. Zhang, On the global wellposedness to the 3-D incompressible anisotropic Navier–Stokes equations, Comm. Math. Phys. 272 (2007) 529–566.

[3] R. Danchin, Global existence in critical spaces for flows of compressible viscous and heat-conductive gases, Arch. Ration. Mech. Anal. 160 (2001) 1–39.

[4] R. Danchin, Fourier Analysis Methods for PDEs, Lecture Notes, 2005, November 14.

[5] P. Degond, Mathematical modelling of microelectronics semiconductor devices, some current topics on nonlinear conservation laws, in: AMS/IP Stud. Adv. Math., vol. 15, Amer. Math. Soc., Providence, RI, 2000, pp. 77–110.

[6] P. Degond, S. Jin, J.-G. Liu, Mach-number uniform asymptotic-preserving gauge schemes for compressible flows, Bull. Inst. Math. Acad. Sin. (N.S.) 2 (2007) 851–892.
[7] D. Donatelli, Local and global existence for the coupled Navier–Stokes–Poisson problem, Quart. Appl. Math. 61 (2003) 345–361.
[8] D. Donatelli, P. Marcati, A quasineutral type limit for the Navier–Stokes–Poisson system with large data, Nonlinearity 21 (2008) 135–148.
[9] B. Ducomet, Some stability results for reactive Navier–Stokes–Poisson systems, in: Evolution Equations: Existence, Regularity and Singularities, Warsaw, 1998, in: Banach Center Publ., vol. 52, Polish Acad. Sci., Warsaw, 2000, pp. 83–118.
[10] B. Ducomet, E. Feireisl, H. Petzeltova, I.S. Skraba, Global in time weak solution for compressible barotropic self-gravitating fluids, Discrete Contin. Dyn. Syst. 11 (2004) 113–130.
[11] B. Ducomet, A. Zlotnik, Stabilization and stability for the spherically symmetric Navier–Stokes–Poisson system, Appl. Math. Lett. 18 (2005) 1190–1198.
[12] Q.-C. Ju, F.-C. Li, H.-L. Li, The quasineutral limit of Navier–Stokes–Poisson system with heat conductivity and general initial data, in press.
[13] H.-L. Li, A. Matsumura, G.-J. Zhang, Optimal decay rate of the compressible Navier–Stokes–Poisson system in R^3, preprint, 2008.
[14] T. Kobayashi, T. Suzuki, Weak solutions to the Navier–Stokes–Poisson equations, preprint, 2004.
[15] M. Paicu, Equation anisotrope de Navier–Stokes dans des espaces critiques, Rev. Mat. Iberoamericana 21 (2005) 179–235.
[16] J. Peetre, New Thoughts on Besov Spaces, Duke Univ. Math. Ser., vol. 1, Duke Univ. Press, Durham, NC, 1976.
[17] S. Wang, S. Jiang, The convergence of the Navier–Stokes–Poisson system to the incompressible Euler equations, Comm. Partial Differential Equations 31 (2006) 571–591.
[18] Y.H. Zhang, Z. Tan, On the existence of solutions to the Navier–Stokes–Poisson equations of a two-dimensional compressible flow, Math. Methods Appl. Sci. 30 (2007) 305–329.